In general: $T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d \qquad T(1) = c$

Master Theorem for Recurrences

$\frac{a}{b^d} < 1 \rightarrow T(n) = O(n^d)$

$\frac{a}{b^d} = 1 \rightarrow T(n) = O(n^d \log n)$

$\frac{a}{b^d} > 1 \rightarrow T(n) = O(n^{\log_b a})$

proof: Let us analyze the tree of work.

$$c \cdot n^d$$

$$c \cdot \left(\frac{n}{b}\right)^d \quad \ldots \quad c \cdot \left(\frac{n}{b}\right)^d$$

$$c \cdot \left(\frac{n}{b^2}\right)^d \quad \ldots$$

$$\vdots$$

$$c$$

problem size	work/problem	# of problems	total work
n	$c \cdot n^d$	1	$c \cdot n^d$
n/b	$c \cdot (n/b)^d$	a	$(a/b^d) \cdot c \cdot n^d$
n/b^2	$c \cdot (n/b^2)^d$	a^2	$(a/b^d)^2 \cdot c \cdot n^d$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n/b^i	$c \cdot (n/b^i)^d$	a^i	$(a/b^d)^i \cdot c \cdot n^d$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$c \cdot 1$	$a^{\log_b n}$	$(a/b^d)^{\log_b n} \cdot c \cdot n^d$

$$c \cdot n^d \cdot \left(1 + \frac{a}{b^d} + \cdots + \left(\frac{a}{b^d}\right)^{\log_b n}\right)$$

We are left to bound $c \cdot n^d \cdot \left(1 + \frac{a}{b^d} + \cdots + \left(\frac{a}{b^d}\right)^{\log_b n}\right)$ in the three cases.

Recall: $1 + p + p^2 + \cdots + p^k = \dfrac{p^{k+1} - 1}{p - 1}$

① root-heavy: if $\frac{a}{b^d} < 1$ (ie, $d > \log_b a$) then

$$T(n) = c \cdot n^d \cdot \frac{1 - \left(\frac{a}{b^d}\right)^{\log_b n + 1}}{1 - \frac{a}{b^d}} = O(n^d)$$

② balanced: if $\frac{a}{b^d} = 1$ (ie, $d = \log_b a$) then

$$T(n) = c \cdot n^d \cdot \underbrace{(1 + \cdots + 1)}_{\log_b n + 1} = O(n^d \log_b n) = O(n^d \log n)$$

③ leaf-heavy: if $\frac{a}{b^d} > 1$ (ie, $d < \log_b a$) then

$$T(n) = c \cdot n^d \cdot \frac{\left(\frac{a}{b^d}\right)^{\log_b n + 1} - 1}{\frac{a}{b^d} - 1} = O\left(n^d \left(\frac{a}{b^d}\right)^{\log_b n}\right) = O\left(n^d \frac{n^{\log_b a}}{n^d}\right) = O\left(n^{\log_b a}\right)$$

Examples: $T(n) = 4T\left(\frac{n}{2}\right) + O(n) \Rightarrow a = 4, b = 2, d = 1 \Rightarrow \frac{a}{b^d} = \frac{4}{2^1} > 1 \Rightarrow O\left(n^{\log_b a}\right) = O(n^2)$

$T(n) = 3T\left(\frac{n}{2}\right) + O(n) \Rightarrow a = 3, b = 2, d = 1 \Rightarrow \frac{a}{b^d} = \frac{3}{2} > 1 \Rightarrow O\left(n^{\log_b a}\right) = O\left(n^{\log_2 3}\right).$