Exercise I, Algorithms 2024-2025

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually.

Basic Algorithms

1 (Based on Exercise 2.1-3 in the book) Consider the **searching problem:**

Input: A sequence of n numbers $A = \langle a_1, a_2, \ldots, a_n \rangle$ and a value v.

Output: An index i such that $v = A[i]$ or the special value NIL if v does not appear in A.

1a Which of the following are instances of the problem and what are the correct outputs for those instances?

$$\langle 1, 5, 3, 4 \rangle \text{ and a value } v$$
$$\langle 3, 8, 3, 4, 50, 47 \rangle \text{ and a value } 50$$
$$\langle 3, 8, 3, 4, 49, 47 \rangle \text{ and a value } 9$$

1b Write pseudocode for **linear search**, which scans through the sequence, looking for v. Using a loop invariant, prove that your algorithm is correct. Make sure that your loop invariant fulfills the three necessary properties (Initialization, Maintenance, Termination).

1c Analyze the worst-case running time of your algorithm in terms of Θ-notation.[1]

Asymptotics

2 Show that for any real constants a and b, where $b > 0$,

$$(n + a)^b = \Theta(n^b).$$

3 Simplify and arrange the following functions in increasing order according to asymptotic growth.

$$3^N, \sqrt{4^N}, \log^2 N, 2^{N \log_2 N}, \sqrt{N}, N^2, \log N, 20N, N!, (N/e)^N$$

4 Express the following functions in terms of Θ-notation:

$$3^N + 2^N, \sqrt{3^N + 4^N}, \log^2(N^3 + 300N^2), \log \binom{N}{2}$$

[1] We may not have had the time to cover runtime analysis in the first lecture. In that case it may be hard to answer this question if you have not seen runtime analysis before.

Proof by Induction

5 Prove the following equalities using induction:

5a $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

5b The Fibonacci series is recursively defined for $n \geq 1$ by

$$f_1 = 1, f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, n \geq 2.$$

Show the Binet formula:

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] \qquad \text{for all } n \geq 1$$

Hint: $\frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2$ and $\frac{3-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2$

6 (*) What is wrong with the following proof that all horses have the same color?

Let $P(n)$ be the proposition that all the horses in a set of n horses are the same color.

Base case: Clearly, $P(1)$ is true.

Now assume that $P(n)$ is true. That is, assume that all the horses in any set of n horses are the same color. Consider any $n+1$ horses; number these as horses $1, 2, 3, ..., n, n+1$. Now the first n of these horses all must have the same color, and the last n of these must also have the same color. Since the set of the first n horses and the set of the last n horses overlap, all $n+1$ must be the same color. This shows that $P(n+1)$ is true and finishes the proof by induction.

A Refresh "Proof by Induction"

Main principles. Recall that the principle of weak induction is

1. The statement is true for a base case (say an integer b).
2. Any time the statement is true for $n \geq b$, one can show that the statement is true for $n+1$.

Under these conditions, the statement is true for all integers $\geq b$.

Similarly, the principle of strong induction is:

1. The statement is true for one or several base cases (say integers $b, b+1, \ldots, b+i$).
2. Any time the statement is true for all integers in $[b, n]$ for some $n \geq b+i$, one can show that the statement is true for $n+1$.

Under these conditions, the statement is true for all integers $\geq b$.

This looks rather technical so let us clarify these concepts with two examples[2]

Example A.1 (weak induction) *Prove by induction that for all positive integers $n \in \mathbb{Z}_+$*

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}. \tag{1}$$

Proof. We will prove (1) by weak induction.

Base case: When $n = 1$, the left side of (1) is $1/(1 \cdot 2) = 1/2$, and the right side is $1/2$, so both sides are equal and (1) is true for $n = 1$.

Induction step: Let $k \in \mathbb{Z}_+$ be given and suppose (1) is true for $n = k$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \textit{(by induction hypothesis)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \quad \textit{(by algebra)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \quad \textit{(by algebra)} \\
&= \frac{k+1}{k+2}.
\end{aligned}$$

Thus, (1) holds for $n = k+1$, and the proof of the induction step is complete.

Conclusion: By the principle of (weak) induction, (1) is true for all $n \in \mathbb{Z}_+$. □

Example A.2 (strong induction) *The fibonacci series is recursively defined for $n \geq 1$ by*

$$f_1 = 1, f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, n \geq 2.$$

We have for all $n \in \mathbb{Z}_+$ that

$$f_n \geq (3/2)^{n-2}. \tag{2}$$

[2]These examples are taken from *http://www.math.uiuc.edu/ hildebr/213/inductionsampler.pdf*. See that page for many more examples.

Proof. **Base cases:** When $n = 1$, the left side of (2) is $f_1 = 1$, and the right side is $(3/2)^{-1} = 2/3$, so (2) holds for $n = 1$. When $n = 2$, the left side of (2) is $f_2 = 1$ and the right side is $(3/2)^0 = 1$, so both sides are equal and (2) is true for $n = 2$.

Induction step: Let $k \geq 2$ be given and suppose (2) is true for all $n = 1, 2 \ldots, k$. Then

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} \quad \textit{(by recurrence for } f_n\textit{)} \\ &\geq (3/2)^{k-2} + (3/2)^{k-3} \quad \textit{(by induction hypothesis with } n = k \textit{ and } n = k-1\textit{)} \\ &= (3/2)^{k-1}\left((3/2)^{-1} + (3/2)^{-2}\right) \quad \textit{(by algebra)} \\ &= (3/2)^{k-1}\left(\frac{2}{3} + \frac{4}{9}\right) \\ &= (3/2)^{k-1}\frac{10}{9} \geq (3/2)^{k-1}. \end{aligned}$$

Thus, (2) holds for $n = k + 1$, and the proof of the induction step is complete.

Conclusion: By the principle of strong induction, it follows that (2) is true for all $n \in \mathbb{Z}_+$.

□

Remarks: Number of base cases: Since the induction step involves the cases $n = k$ and $n = k - 1$, we can carry out this step only for values $k \geq 2$ (for $k = 1, k - 1$ would be 0 and out of range). This in turn forces us to include the cases $n = 1$ and $n = 2$ in the base step. Such multiple base cases are typical in proofs involving recurrence sequences. For a three term recurrence we would need to check three initial cases, $n = 1, 2, 3$, and in the induction step restrict k to values 3 or greater.