Dynamical Systems : Extended Summary Allowed at Final Exam

Julien Herzen and Patrick Thiran

1. Basic Notions

Evolution equations:

Continuous time	Discrete time	
$\frac{dx}{dt} = F(x(t), u(t))$	$x(t+1) = F(x(t), u(t))$	(state equation)
$y(t) = G(x(t), u(t))$	$y(t) = G(x(t), u(t))$	(output equation)

where $F : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $G : \mathbb{R}^{n+m} \to \mathbb{R}^p$, and

$$u(t) \in \mathcal{T} \subseteq \mathbb{R}^m \quad \text{input space} \tag{1}$$
$$x(t) \in \Omega \subseteq \mathbb{R}^n \quad \text{state space} \tag{2}$$
$$y(t) \in \Theta \subseteq \mathbb{R}^p \quad \text{output space} \tag{3}$$

Definition 1.1 (solution). *A solution of the system is a function $x : \mathcal{T} \to \Omega$.*

Definition 1.2 (trajectory of a solution). *Given a solution $x(t)$, the set $\{(t, x(t)) \mid t \in \mathcal{T}\}$ is the trajectory of the solution.*

Definition 1.3 (orbit of a solution). *Given a solution $x(t)$, the set $\{x(t) \mid t \in \mathcal{T}\}$ is the orbit of the solution.*

Theorem 1.1 (Unique Solution). *For each initial state $x(0) \in \mathbb{R}^n$, a discrete-time dynamical system admits exactly one solution $x : \mathbb{N} \to \mathbb{R}^n$. If $F(x(t), u(t))$ is continuous and locally Lipschitz with respect to x, then for each initial state $x(0) \in \mathbb{R}^n$, a continuous-time dynamical system admits exactly one solution $x : \mathbb{R}_+ \to \mathbb{R}^n$.*

It may happen that a solution exists only up to a finite positive (resp. negative) *escape* time t_1, in which case

$$\|x(t)\| \xrightarrow[t \to t_1]{} \infty \tag{4}$$

Definition 1.4 (fixed / equilibrium point)**.** *In continuous time, a state* $\bar{x}$ *that satisfies* $\frac{df}{dx}(\bar{x}) = 0$ *is called an* equilibrium point *of the system.*
In discrete time, a state $\bar{x}$ *that satisfies* $\bar{x} = F(\bar{x})$ *is called a* fixed point *of the system.*

Definition 1.5 (flow of a system)**.** *The flow of a system consists in all the solutions considered simultaneously. It is a function* $\Phi : \mathcal{T} \times \Omega \to \Omega$, *defined by* $\Phi(t, x(0)) \mapsto x(t)$. *It can be represented graphically by drawing the trajectories of a few solutions.*

Definition 1.6 (transient / asymptotic behaviors)**.** *The* transient behavior *is the behavior of the system for small time* t, *when it strongly depends on the input signal. The* asymptotic behavior *is the behavior of the system for large time* t, *when the system is in steady state. Mathematically, it is the behavior for* $t \to \infty$.

Definition 1.7 (unique asymptotic behavior)**.** *A system has* unique asymptotic behavior *if for any two solutions* $x(t)$ *and* $\tilde{x}(t)$ *we have:*

$$\|x(t) - \tilde{x}(t)\| \xrightarrow[t\to\infty]{} 0. \tag{5}$$

Definition 1.8 (periodic solution)**.** *A solution is* periodic *if there exists a time* T *such that* $x(t+T) = x(t)\ \forall t$.

Definition 1.9 (autonomous system)**.** *A system is* autonomous *if it has no input signal.*

Definition 1.10 (Invariant Set)**.** *(i) A set* $\mathcal{S} \subseteq \mathbb{R}^n$ *is forward invariant if for any solution* x *such that at some time* $t \in \mathcal{T}$ $x(t) \in \mathcal{S}$, *it follows that* $x(t') \in \mathcal{S}$ *for all* $t' \geq t$.

(ii) A set $\mathcal{S} \subseteq \mathbb{R}^n$ *is backward invariant if for any solution* x *such that at some time* $t \in \mathcal{T}$ $x(t) \in \mathcal{S}$, *it follows that* $x(t') \in \mathcal{S}$ *for all* $t' \leq t$.

(iii) A set $\mathcal{S} \subseteq \mathbb{R}^n$ *is invariant if it is both forward and backward invariant.*

Definition 1.11 (α- and ω-limit sets)**.** *Consider a solution* $x(t)$ *of an autonomous dynamical system in* $\mathbb{R}^n$, *defined for* $\mathcal{T} = \mathbb{R}$ *or* $\mathcal{T} = \mathbb{Z}$ *and unique for a given* $x(0)$.

(i) The ω*-limit set of the solution is the set of points* $\xi \in \mathbb{R}^n$ *such that there exists a sequence of times* $t_1 < t_2 < \ldots < t_i < \ldots$ *with* $t_i \to \infty$ *when* $i \to \infty$, *such that*

$$\lim_{i\to\infty} x(t_i) = \xi. \tag{6}$$

(ii) The α*-limit set of the solution is the set of points* $\xi \in \mathbb{R}^n$ *such that there exists a sequence of times* $t_1 > t_2 > \ldots > t_i > \ldots$ *with* $t_i \to -\infty$ *when* $i \to \infty$, *such that (6) holds.*

Theorem 1.2. *Let* $x(t)$ *be a solution of the discrete-time system* $x(t+1) = F(x(t))$, *where* F *is continuous, invertible and its inverse is also continuous. Then*

(i) If the solution is bounded for $t \to +\infty$, *its* ω*-limit set is compact (bounded and closed), non-empty and invariant.*

(ii) If the solution is bounded for $t \to -\infty$, *its* α*-limit set is compact, non-empty and invariant.*

Theorem 1.3. *Let* $x(t)$ *be a solution of the continuous-time system* $dx/dt(t) = F(x(t))$.

(i) If the solution is bounded for $t > 0$, *its* ω*-limit set is compact (bounded and closed), non-empty and connected.*

(ii) If the solution is bounded for $t < 0$, *its* α*-limit set is compact, non-empty and connected.*

Definition 1.12 (Attractor). *A non-empty compact (bounded and closed) set of the state space $\mathcal{A} \subseteq \Omega$ is an attractor of the system, if the following conditions hold:*

(i) $\mathcal{A}$ is forward invariant.

(ii) There exists a neighborhood $\mathcal{U}$ of $\mathcal{A}$, which is an open set $\mathcal{U} \supset \mathcal{A}$ such that all solutions starting in $\mathcal{U}$ converge to $\mathcal{A}$ as $t \to +\infty$.

(iii) There is no proper non-empty compact subset of $\mathcal{A}$ that has properties (i) and (ii).

Remark 1.1. A solution $x(t)$ is said to converge to a set A if the distance between $x(t)$ and A converges to 0 as $t \to \infty$. The distance between a point ξ and a set S is defined as $d(\xi, S) = \inf_{\eta \in S} d(\xi, \eta)$

2. Linear Systems

2.1. Systems and General Solutions

Discrete time:

$$x(t+1) = Ax(t) + Bu(t) \tag{7}$$
$$y(t) = Cx(t) + Du(t) \tag{8}$$

Solutions:

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{t-\tau-1} Bu(\tau) \tag{9}$$
$$y(t) = CA^t x(0) + \left(C \sum_{\tau=0}^{t-1} A^{t-\tau-1} Bu(\tau) \right) + Du(t) \tag{10}$$

Continuous time:

$$\frac{dx}{dt} = Ax(t) + Bu(t) \tag{11}$$
$$y(t) = Cx(t) + Du(t) \tag{12}$$

Solutions:

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \tag{13}$$
$$y(t) = Ce^{At} x(0) + \left(C \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \right) + Du(t) \tag{14}$$

The free solution (when $u(t) = 0$) can be represented as

$$\begin{aligned} x(t) &= \phi(t) x(0) \\ y(t) &= C\phi(t) x(0), \end{aligned}$$

where $\phi(t) = e^{At}$ for continuous-time systems and $\phi(t) = A^t$ for discret time for discrete-time systems.

2.2. Matrix Exponential

The matrix exponential of tA is

$$e^{tA} = I_n + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots,$$

where I_n is the $n \times n$ identity matrix. If J is the Jordan block decomposition of A, $e^{tA} = Se^{tJ}S^{-1}$ where

$$e^{tJ} = \begin{bmatrix} e^{tJ_1} & 0 & 0 & \dots & 0 \\ 0 & e^{tJ_2} & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & \ddots & 0 \\ 0 & \dots & & 0 & e^{tJ_q} \end{bmatrix}$$

with

$$e^{tJ_i} = \begin{bmatrix} e^{t\lambda_i} & te^{t\lambda_i} & \frac{t^2}{2}e^{t\lambda_i} & \dots & \frac{t^{n_i-1}}{(n_i-1)!}e^{t\lambda_i} \\ 0 & e^{t\lambda_i} & te^{t\lambda_i} & \frac{t^2}{2}e^{t\lambda_i} & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & 0 & e^{t\lambda_i} & te^{t\lambda_i} \\ 0 & \dots & & 0 & e^{t\lambda_i} \end{bmatrix},$$

and where λ_i is an eigenvalue of A and where the columns of S are the corresponding (generalised) eigenvectors.

2.3. Stability

Definition 2.1 (Stability of a linear system)**.** *(i) A linear system is* asymptotically stable *if for all $x_0 \in \Omega$ we have*

$$\|\phi(t)x_0\| \xrightarrow[t\to\infty]{} 0. \tag{15}$$

(ii) A linear system is stable *if for all $x_0 \in \Omega$ there is a constant $C > 0$ such that for all $t \in \mathcal{T}$*

$$\|\phi(t)x_0\| \leq C. \tag{16}$$

(iii) A linear system is weakly unstable *if it not stable and if for all $x_0 \in \Omega$ there exist $C > 0$ and $n > 0$ such that for all $t \in \mathcal{T}$*

$$\|\phi(t)x_0\| \leq Ct^n. \tag{17}$$

(iv) A linear system is strongly unstable *if it neither stable nor weakly unstable.*

Theorem 2.1. *If λ_i, $1 \leq i \leq n$, are the eigenvalues of its state matrix A, then a linear autonomous continuous-time system is*

- *asymptotically stable if all eigenvalues λ_i satisfy $\mathfrak{Re}(\lambda_i) < 0$ for $1 \leq i \leq n$,*
- *stable if all eigenvalues λ_i, for $1 \leq i \leq n$, satisfy either $\mathfrak{Re}(\lambda_i) < 0$, or $\mathfrak{Re}(\lambda_i) = 0$ and the corresponding Jordan block J_i is of dimension 1 ($r_i = 1$, $J_i = \lambda_i$),*

- *weakly unstable if all eigenvalues* λ_i *satisfy* $\mathfrak{Re}(\lambda_i) \leq 0$ *for* $1 \leq i \leq n$*, and if there is at least one eigenvalue* λ_i *with* $\mathfrak{Re}(\lambda_i) = 0$ *such that the corresponding Jordan block* J_i *is of dimension higher than 1 (*$r_i > 1$*),*
- *strongly unstable if* $\mathfrak{Re}(\lambda_i) > 0$ *for at least one eigenvalue* λ_i*,* $1 \leq i \leq n$*.*

Theorem 2.2. *If* λ_i*,* $1 \leq i \leq n$*, are the eigenvalues of its state matrix* A*, then a linear autonomous discrete-time system is*

- *asymptotically stable if all eigenvalues* λ_i *satisfy* $|\lambda_i| < 1$ *for* $1 \leq i \leq n$*,*
- *stable if all eigenvalues* λ_i*, for* $1 \leq i \leq n$*, satisfy either* $|\lambda_i| < 1$*, or* $|\lambda_i| = 1$ *and the corresponding Jordan block* J_i *is of dimension 1 (*$r_i = 1$*,* $J_i = \lambda_i$*),*
- *weakly unstable if all eigenvalues* λ_i *satisfy* $|\lambda_i| \leq 1$ *for* $1 \leq i \leq n$*, and if there is at least one eigenvalue* λ_i *with* $|\lambda_i| = 1$ *such that the corresponding Jordan block* J_i *is of dimension higher than 1 (*$r_i > 1$*),*
- *strongly unstable if* $|\lambda_i| > 1$ *for at least one eigenvalue* λ_i*,* $1 \leq i \leq n$*.*

2.4. BIBO Stability

The state and output equations in the Laplace (frequency) domain of a continuous-time system are

$$\begin{aligned} X(s) &= (sI_n - A)^{-1}x(0) + (sI_n - A)^{-1}BU(s) \\ Y(s) &= C(sI_n - A)^{-1}x(0) + (C(sI_n - A)^{-1}B + D)U(s) \end{aligned}$$

where

$$U(s) = \int_0^\infty u(t)e^{-st}dt$$

is the Laplace transform of the input signal $u(t)$ (and similarly, $X(s), Y(s)$ are the Laplace transforms of $x(t), y(t)$, respectively). The transfer matrix of the zero-state system (i.e. with $x(0) = 0$) is $H(s) = C(sI_n - A)^{-1}B + D$.

The z-transforms of the state and output equations of a discrete-time system are

$$\begin{aligned} X(z) &= (zI_n - A)^{-1}x(0) + (zI_n - A)^{-1}BU(z) \\ Y(z) &= C(zI_n - A)^{-1}x(0) + (C(zI_n - A)^{-1}B + D)U(z) \end{aligned}$$

where

$$U(z) = \sum_{n=0}^{\infty} u(n)z^{-n}$$

is the z-transform of the input signal $u(n)$ (and similarly, $X(z), Y(z)$ are the z-transforms of $x(n), y(n)$, respectively). The transfer matrix of the zero-state system is $H(z) = C(zI_n - A)^{-1}B + D$.

Definition 2.2 (B.I.B.O. stability of a linear system)**.** *A linear system is* B.I.B.O. stable *if and only if for all* $u \in \Gamma$ *such that*

$$\|u(t)\| \leq u_M$$

for some finite u_M *and for all* $t \in \mathcal{T}$*, there is a finite constant* K *such that*

$$\|y(t)\| \leq Ku_M$$

for all $t \in \mathcal{T}$*.*

Theorem 2.3. *A linear time-invariant system with impulse response $h(t)$ is BIBO stable if and only if there is some finite h_M such that*

$$\int_0^\infty \|h(\tau)\| d\tau \leq h_M. \tag{18}$$

In the previous expressions, the norm is the infinite norm. In the multi-dimensional case, if h_{ij} denotes the (i, j)th entry of h, (18) can therefore be replaced by

$$\int_0^\infty \max_{ij} |h_{ij}(\tau)| d\tau \leq h_M.$$

If the transfer function $H(s)$ can be expressed as a rational fraction in s, then the roots of its denominator are called in control theory the *poles* of the system, they are the values of s for which $H(s)$ is infinite. Poles are always natural frequencies of the system.

Theorem 2.4. *A linear time-invariant system with a transfer function $H(s)$ which is a rational function of s is BIBO stable if and only if all the poles of every entry of $H(s)$ have strictly negative real parts.*

3. Stability of Nonlinear Systems

From now on, we only consider autonomous systems either in continuous time

$$\frac{dx}{dt}(t) = F(x(t)) \tag{19}$$

or in discrete time

$$x(t+1) = F(x(t)) \tag{20}$$

with $F : \mathbb{R}^n \to \mathbb{R}^n$. We sometimes use $\dot{x}$ as a short-hand for dx/dt.

3.1. Large-scale Notions of Stability/instability

Definition 3.1 (Large-scale stability)**.** *(i) An autonomous system has* bounded solutions *if for each $x_0 \in \mathbb{R}^n$ there is a constant $B > 0$ such that for all $t \geq 0$*

$$\|x(t)\| \leq B$$

where $x(t)$ is the solution with initial condition $x(0) = x_0$.

(ii) An autonomous system has asymptotically uniformly bounded solutions *if there is a constant $B > 0$ such that for each $x_0 \in \mathbb{R}^n$, there is a finite time $T \geq 0$ such that for all $t \geq T$*

$$\|x(t)\| \leq B$$

where $x(t)$ is the solution with initial condition $x(0) = x_0$.

3.1.1. Lyapunov Functions

For a continuous-time system $\dot{x} = F(x)$, W is *non-increasing along trajectories*, if for any solution $x(t)$,

$$\dot{W}(x) = \nabla_x^T W(x) F(x) \leq 0, \tag{21}$$

where T denotes transposition and where $\nabla_x W(x)$ is the gradient of W at point x:

$$\nabla_x W(x) = \begin{bmatrix} \frac{\partial W}{\partial x_1}(x_1, \ldots, x_n) \\ \frac{\partial W}{\partial x_2}(x_1, \ldots, x_n) \\ \vdots \\ \frac{\partial W}{\partial x_n}(x_1, \ldots, x_n) \end{bmatrix}.$$

For a discrete-time system $x(t+1) = F(x(t))$, (21) is replaced by

$$W(x(t+1)) - W(x(t)) = W(F(x(t)) - W(x(t)) \leq 0. \tag{22}$$

If we replace the inequality sign in (21) and (22) by a strict inequality, W is *decreasing along trajectories*. In the following theorem, the Kth-level set of a function $W : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{L}_K = \{x \in \mathbb{R}^n \mid W(x) \leq K\}$.

Theorem 3.1. *Suppose there is a function $W : \mathbb{R}^n \to \mathbb{R}$ such that*

- *for discrete time systems, W is continuous and for continuous time systems, W is continuously differentiable,*
- *$W(x) \geq 0$ for all $x \in \mathbb{R}^n$,*
- *the level sets $\mathcal{L}_K = \{x \in \mathbb{R}^n \mid W(x) \leq K\}$ are bounded for all $K > 0$.*

(i) Suppose in addition that

- *W is non-increasing along trajectories,*

then the system has bounded solutions.
(ii) Suppose in addition that there is a constant $E > 0$ such that

- *W is decreasing along trajectories as long as $W(x(t)) \geq E$ (i.e. for $x(t) \notin \mathcal{L}_E$)*
- *for a discrete-time system, the level set $\mathcal{L}_E$ is forward invariant,*

then the system has asymptotically uniformly bounded solutions.

3.1.2. Hamiltonian Systems

Hamiltonian systems have an even dimension: $n = 2r$ for some $r \in \mathbb{N}$, andare described by a function $H : \mathbb{R}^{2r} \to \mathbb{R}$, the Hamilton function or *Hamiltonian*, from which the state equation are derived as follows, for $1 \leq i \leq r$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{23}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{24}$$

The first r arguments $q_1, \dots, q_r$ of H are called generalized coordinates and the last r arguments $p_1, \dots, p_r$ of H are called generalized momenta. Together, they constitute the $2r$-dimensional state vector. The fact that along solutions of the system (23), (24) the value of H is constant can be seen easily by computing

$$\begin{aligned} \dot{H}\left(q_1(t), \dots, q_r(t), p_1(t), \dots, p_r(t)\right) &= \sum_{i=1}^{r} \frac{\partial H}{\partial q_i} \dot{q}_i + \sum_{i=1}^{r} \frac{\partial H}{\partial p_i} \dot{p}_i \\ &= -\sum_{i=1}^{r} \dot{p}_i \dot{q}_i + \sum_{i=1}^{r} \dot{q}_i \dot{p}_i = 0. \end{aligned}$$

3.2. Small-scale Notions of Stability/instability

Definition 3.2 (Small-scale stability). *(i) A solution $x^\star : \mathbb{N} \to \mathbb{R}^n$ of a discrete-time autonomous system, or a solution $x^\star : \mathbb{R}_+ \to \mathbb{R}^n$ of a continuous-time system, is* stable, *if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any solution x with*

$$\|x(0) - x^\star(0)\| \leq \delta$$

we have for all $t \geq 0$

$$\|x(t) - x^\star(t)\| \leq \varepsilon.$$

- A solution is unstable *if it is not stable.*

(ii) A solution $x^\star$ is asymptotically stable, *or more precisely locally asymptotically stable, if it is stable, i.e. if it verifies the conditions in item (i) of the definition, and if in addition*

$$\lim_{t \to +\infty} \|x(t) - x^\star(t)\| = 0. \tag{25}$$

The basin of attraction *at time $t = 0$ of an asymptotically stable solution $x^\star$ is the set of all $x_0 \in \mathbb{R}^n$ such that the solution x with initial condition $x(0) = x_0$ satisfies (25).*

(iii) A solution $x^\star$ is globally asymptotically stable *if it is asymptotically stable and if its basin of attraction is the whole space $\mathbb{R}^n$. In this case, actually all solutions are globally asymptotically stable and for any two solutions (25) holds. Thus, this is a property of the system rather than of the solution and the system is said to have a* unique asymptotic behavior*.*

3.2.1. Stability of a fixed point in a discrete-time system

Let $\overline{x}$ be a fixed point of F, i.e. a point such that

$$F(\overline{x}) = \overline{x}. \tag{26}$$

and let $J(\overline{x})$ be the *Jacobian matrix* of F at the fixed point $\overline{x}$:

$$J(\overline{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(\overline{x}) & \frac{\partial F_1}{\partial x_2}(\overline{x}) & \dots & \frac{\partial F_1}{\partial x_n}(\overline{x}) \\ \frac{\partial F_2}{\partial x_1}(\overline{x}) & \frac{\partial F_2}{\partial x_2}(\overline{x}) & \dots & \frac{\partial F_2}{\partial x_n}(\overline{x}) \\ \vdots & & \ddots & \\ \frac{\partial F_n}{\partial x_1}(\overline{x}) & \frac{\partial F_n}{\partial x_2}(\overline{x}) & \dots & \frac{\partial F_n}{\partial x_n}(\overline{x}) \end{bmatrix} \tag{27}$$

A fixed point $\overline{x}$ of a discrete time system is *hyperbolic*, if no eigenvalue of the Jacobian matrix $J(\overline{x})$ lies on the unit circle (i.e. has magnitude equal to 1).

Theorem 3.2. *Let $\overline{x}$ be a hyperbolic fixed point of a discrete time autonomous system given by (20).*

(i) The fixed point $\overline{x}$ is asymptotically stable if and only if all eigenvalues λ_i of the Jacobian matrix $J(\overline{x})$ satisfy $|\lambda_i| < 1$.

(ii) The fixed point $\overline{x}$ is unstable if and only if at least one eigenvalue λ_i of the Jacobian matrix $J(\overline{x})$ satisfies $|\lambda_i| > 1$.

3.2.2. Stability of an equilibrium point in a conitnuous-time system

Let $\overline{x}$ be an equilibrium point, i.e. a point such that

$$F(\overline{x}) = 0. \tag{28}$$

An equilibrium point $\overline{x}$ of a continuous-time system is *hyperbolic*, if no eigenvalue of the Jacobian matrix $J(\overline{x})$ given by (27) lies on the imaginary axis (i.e. has a zero real part).

Theorem 3.3. *Let $\overline{x}$ be a hyperbolic equilibrium point of a continuous-time autonomous system given by (19).*

(i) The equilibrium point $\overline{x}$ is asymptotically stable if and only if all eigenvalues λ_i of the Jacobian matrix $J(\overline{x})$ satisfy $\Re\{\lambda_i\} < 0$.

(ii) The equilibrium point $\overline{x}$ is unstable if and only if at least one eigenvalue λ_i of the Jacobian matrix $J(\overline{x})$ satisfies $\Re\{\lambda_i\} > 0$.

3.2.3. Nature of the flow in a neighborhood of an equilibrium/fixed point

If a fixed/equilibrium point is hyperbolic, the notions of eigenspaces known for linear systems are extended as follows.

Definition 3.3. *The* stable manifold W^s *of an equilibrium/fixed point $\overline{x}$ is the set*

$$W^s = \left\{ x_0 \mid x(t) \textit{ is a solution starting at } x(0) = x_0 \textit{ such that } \lim_{t\to\infty} x(t) = \overline{x} \right\}.$$

The unstable manifold W^u *of an equilibrium/fixed point $\overline{x}$ is the set*

$$W^u = \left\{ x_0 \mid x(t) \textit{ is a solution starting at } x(0) = x_0 \textit{ such that } \lim_{t\to-\infty} x(t) = \overline{x} \right\}.$$

Let $\overline{x}$ be an hyperbolic equilibrium/fixed point and V^s (respectively, V^u) be the contracting (resp., expanding) linear subspace of the linearization of the system at $\overline{x}$. The manifolds of $\overline{x}$ enjoy the following properties:

- The stable manifold W^s of $\overline{x}$ is an invariant set. In a neighborhood of $\overline{x}$, W^s is a surface of the same dimension as V^s and it is tangent to V^s at $\overline{x}$.
- The unstable manifold W^u of $\overline{x}$ is an invariant set. In a neighborhood of $\overline{x}$, W^u is a surface of the same dimension as V^u and it is tangent to V^u at $\overline{x}$.

3.2.4. Estimation of the basins of attraction of asymptotically stable fixed/equilibrium points

Lyapunov functions were introduced in Section 3.1.1. The strict Kth-level set of a function $W : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{U}_K = \{x \in \mathbb{R}^n \mid W(x) < K\}$.

Theorem 3.4. *Let $\overline{x}$ be an equilibrium/fixed point. Suppose there is a function $W : \mathbb{R}^n \to \mathbb{R}$ and a constant $E > 0$ such that*

- *for discrete time systems, W is continuous and for continuous time systems, W is continuously differentiable,*
- *$W(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{\overline{x}\}$, and $W(\overline{x}) = 0$,*
- *the strict level set $\mathcal{U}_E = \{x \in \mathbb{R}^n \mid W(x) < E\}$ is bounded,*
- *W is non-increasing along trajectories as long as $W(x(t)) < E$ (i.e. for $x(t) \in \mathcal{U}_E$),*
- *W is decreasing along trajectories as long as $W(x(t)) < E$ and $x(t) \neq \overline{x}$ (i.e. for $x(t) \in \mathcal{U}_E \setminus \{\overline{x}\}$).*

Then U_E is contained in the basin of attraction of $\overline{x}$.

Corollary 3.1. *Let $\overline{x}$ be an equilibrium/fixed point. Suppose there is a function $W : \mathbb{R}^n \to \mathbb{R}$ such that*

- *for discrete time systems, W is continuous and for continuous time systems, W is continuously differentiable,*
- *$W(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{\overline{x}\}$, and $W(\overline{x}) = 0$,*
- *the strict level sets $\mathcal{U}_K = \{x \in \mathbb{R}^n \mid W(x) < K\}$ are bounded for all $K > 0$,*
- *W is non-increasing along trajectories,*
- *W is decreasing along trajectories as long as $x(t) \neq \overline{x}$ (i.e. for $x(t) \in \mathbb{R}^n \setminus \{\overline{x}\}$).*

Then $\overline{x}$ is globally asymptotically stable.

3.2.5. Gradient Systems

Definition 3.4 (Gradient System)**.** *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function. Then the gradient system generated by V is defined by the state equation*

$$\dot{x} = -\nabla_x V(x). \tag{29}$$

It follows from this definition that along any solution $x(t)$ of the system, V is non-increasing, yielding the following properties.

- If $\overline{x}$ is the unique equilibrium point of the system, if $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and if all strict level sets of V are bounded, then Corollary 3.1 yields that $\overline{x}$ is globally asymptotically stable: all solutions converge to $\overline{x}$.
- Gradient systems have no periodic solutions other than equilibrium points.
- If all equilibrium points are isolated, and if all strict level sets of V are bounded, then all solutions converge to an equilibrium point (stable or unstable, possibly different equilibrium points for different solutions).

3.2.6. Stability of a periodic solution of a discrete-time system

Let ξ be a T-periodic solution ξ, which is therefore such that

$$\xi(t+T) = \xi(t) \quad \text{for all} \;\; t \in \mathcal{T}. \tag{30}$$

For discrete-time systems, $\mathcal{T} = \mathbb{N}$, for continuous-time systems, $\mathcal{T} = \mathbb{R}^+$.

Theorem 3.5. *Let ξ be a T-periodic solution of a discrete-time autonomous system given by (20), with $T \in \mathbb{N}^*$. Then $\xi(0), \xi(1), \ldots \xi(T-1)$ are fixed points of the mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$G(x) = F(F(F(\ldots(F(x))))) = F^{(T)}(x). \tag{31}$$

Conversely, to any fixed point of G corresponds a T-periodic solution of the system given by the iterations of F. The stability properties of the T-periodic solution ξ are the same as the stability properties of the corresponding fixed points of (31).

Variational Equation. Let $x^\star(t)$ be a particular solution of a discrete-time autonomous system given by (20), with initial condition $x^\star(0)$ (Here we consider a T-periodic solution, but the variational equation can be written with respect to any such solution). If a solution x starts at an initial condition $x(0)$ close to $x^\star(0)$, and if we define its increment with respect to the solution $x^\star$ by

$$\Delta x = x - x^\star, \tag{32}$$

then

$$\Delta x(t) = \Phi(t, x(0)) - \Phi(t, x^\star(0)). \tag{33}$$

The first order approximation to the increment reads

$$\Delta x(t) = M(t) \Delta x(0) \tag{34}$$

where $M(t)$ is the Jacobian matrix of Φ with respect to x_0 at the point $x^\star(0)$ and for a given time t, which is

$$M(t) = \begin{bmatrix} \frac{\partial \Phi_1}{\partial x_{01}}(t, x^\star(0)) & \frac{\partial \Phi_1}{\partial x_{02}}(t, x^\star(0)) & \ldots & \frac{\partial \Phi_1}{\partial x_{0n}}(t, x^\star(0)) \\ \frac{\partial \Phi_2}{\partial x_{01}}(t, x^\star(0)) & \frac{\partial \Phi_2}{\partial x_{02}}(t, x^\star(0)) & \ldots & \frac{\partial \Phi_2}{\partial x_{0n}}(t, x^\star(0)) \\ \vdots & & \ddots & \\ \frac{\partial \Phi_n}{\partial x_{01}}(t, x^\star(0)) & \frac{\partial \Phi_n}{\partial x_{02}}(t, x^\star(0)) & \ldots & \frac{\partial \Phi_n}{\partial x_{0n}}(t, x^\star(0)) \end{bmatrix}, \tag{35}$$

with $M(0) = I_n$. Now, because of (20), the flow $\Phi(t, x_0)$ verifies

$$\Phi(t+1, x_0) = F(\Phi(t, x_0)).$$

If we differentiate this equation with respect to x_0 at $x^\star(0)$, we obtain

$$M(t+1) = J\left(x^\star(t)\right) M(t) \tag{36}$$

where $J(x^\star(t))$ is the *Jacobian matrix* of F at $x^\star(t)$, given by

$$J\left(x^\star(t)\right) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(x^\star(t)) & \frac{\partial F_1}{\partial x_2}(x^\star(t)) & \ldots & \frac{\partial F_1}{\partial x_n}(x^\star(t)) \\ \frac{\partial F_2}{\partial x_1}(x^\star(t)) & \frac{\partial F_2}{\partial x_2}(x^\star(t)) & \ldots & \frac{\partial F_2}{\partial x_n}(x^\star(t)) \\ \vdots & & \ddots & \\ \frac{\partial F_n}{\partial x_1}(x^\star(t)) & \frac{\partial F_n}{\partial x_2}(x^\star(t)) & \ldots & \frac{\partial F_n}{\partial x_n}(x^\star(t)) \end{bmatrix}$$

Equation (36) is called the *variational equation* of the discrete-time system along the solution $x^\star(t)$. Its solution is

$$M(t) = J\left(x^\star(t-1)\right) J\left(x^\star(t-2)\right) \cdots J\left(x^\star(1)\right) J\left(x^\star(0)\right). \tag{37}$$

The variational equation yields following corollary of Theorem 3.5.

Corollary 3.2. *Let ξ be a T-periodic solution of a discrete-time autonomous system given by (20), with $T \in \mathbb{N}^*$, such that the fixed point $\xi(0)$ of the mapping $G = F^{(T)}$ is hyperbolic. Then the solution is asymptotically stable if and only if the matrix $M(T)$ given by (37), with $x^\star = \xi$, has all its eigenvalues within the unit circle.*

3.2.7. Stability of a periodic solution of a continuous-time system

Variational Equation. Let $x^\star(t)$ be a solution of an with initial condition $x^\star(0)$. In continuous-time, the flow $\Phi(t, x_0)$ verifies

$$\frac{\partial \Phi}{\partial t}(t, x_0) = F(\Phi(t, x_0)).$$

If we differentiate this equation with respect to x_0 at $x^\star(0)$, we obtain the variational equation for the continuous time system along the solution $x^\star$

$$\dot{M}(t) = J\left(x^\star(t)\right) M(t) \tag{38}$$

This is a linear time-dependent differential equation for the matrix function $M(t)$. If we combine (38) with the original system equation (19), we obtain the time-independent nonlinear system of $n + n^2$ differential equations

$$\begin{aligned} \dot{x}^\star &= F\left(x^\star\right) && (39) \\ \dot{M} &= J\left(x^\star\right) M. && (40) \end{aligned}$$

with the initial conditions $x(0)$ and $M(0) = I_n$.

Stability and Floquet Multipliers. Suppose that the system (19) has a T-periodic solution ξ that verifies therefore (30). Let us denote by Γ the orbit of the periodic solution, which is also called a *cycle*.

$$\Gamma = \{x \in \mathbb{R}^n \mid x = \xi(t) \text{ for some } 0 \leq t \leq T\}.$$

Now consider the hyperplane $\mathcal{P}$ in $\mathbb{R}^n$, which intersects Γ perpendicularly at the point $\xi(0)$, i.e., which is orthogonal to the vector $\dot{\xi}(0)$:

$$\mathcal{P} = \left\{x \in \mathbb{R}^n \mid \dot{\xi}^T(0)(x - \xi(0)) = 0\right\}.$$

(Note that the exponent T in the above expression denotes transposition). This plane $\mathcal{P}$ is called the Poincaré section at $\xi(0)$.

In a neighborhood $\mathcal{U}$ of $\xi(0)$ on $\mathcal{P}$ we define the first return map R, or *Poincaré map*, as follows. Let $x_0 \in \mathcal{U}$ and let $x(t)$ be the solution with $x(0) = x_0$. This solution will intersect $\mathcal{P}$ again, after approximately time T, at a point $R(x_0)$. Clearly, $\xi(0)$ is a fixed point of the return map.

Theorem 3.6. *Let ξ be a T-periodic solution of a continuous-time autonomous system given by (19), with $T \in \mathbb{R}^+$, and let Γ be its orbit. Consider the first return map R of the Poincaré section $\mathcal{P}$ through $\xi(0)$. Then the periodic solution ξ is stable if the fixed point $\xi(0)$ of the first return map R is stable. Furthermore, the ω-limit set of any solution x of the system starting sufficiently close to $\xi(0)$ is Γ.*

Corollary 3.3. *Let ξ be a T-periodic solution of a continuous-time autonomous system given by (19), with $T \in \mathbb{R}^+$. Consider the first return map R of the Poincaré section $\mathcal{P}$ through $\xi(0)$. Suppose that all eigenvalues λ_i of the Jacobian matrix of R at $\xi(0)$ satisfy $|\lambda_i| < 1$ for all $1 \leq i \leq n-1$. Then the periodic solution ξ is stable.*

Lemma 3.1. *The linearization of the first return map R is $[P \cdot M(T)]_{\mathcal{P}}$ where $M(t)$ is the solution of the variational equations around the periodic solution $\xi(t)$, and where P is the orthogonal projection onto the Poincaré section $\mathcal{P}$, taking $\xi(0)$ as the origin.*

The eigenvalues of $M(T)$ are called the *Floquet multipliers*. They are related to the eigenvalues of the return map by the following theorem.

Lemma 3.2. *Let ξ be a T-periodic solution of a continuous-time autonomous system given by (19), with $T \in \mathbb{R}^+$. Consider the first return map R of the Poincaré section $\mathcal{P}$ through $\xi(0)$ and the solution $M(t)$ of the variational equations (39) and (40) along $\xi(t)$. The eigenvalues of the Jacobian matrix of R at $\xi(0)$ are $\lambda_1, \ldots, \lambda_{n-1}$ if and only if the Floquet multipliers, i.e. the eigenvalues of $M(T)$, are $\lambda_1, \ldots, \lambda_{n-1}, 1$.*

Theorem 3.7. *Let ξ be a T-periodic solution of a continuous-time autonomous system given by (19), with $T \in \mathbb{R}^+$, and let Γ be its orbit. Consider the first return map R of the Poincaré section $\mathcal{P}$ through $\xi(0)$ and the solution $M(t)$ of the variational equations (39) and (40) along $\xi(t)$. If all the Floquet multipliers λ_i, $1 \leq i \leq n-1$ (the eigenvalues of $M(T)$), except one (which is $\lambda_n = 1$) satisfy $|\lambda_i| < 1$, the periodic solution $\xi(t)$ is stable. In addition, any solution starting sufficiently close to $\xi(0)$ converges to a time-shifted version of the periodic solution $\xi(t)$, i.e. its ω-limit set is Γ.*

4. Periodic Solutions in Planar Systems

The orbit of a non-trivial periodic solution, which we denote by Γ, is called a *cycle*. It is called a *limit cycle* if it is the α-limit set or ω-limit set of some solution whose initial condition is not on the cycle. We are looking for non trivial periodic solutions of 2-dim. autonomous systems continuous-time systems.

4.1. Absence of a periodic solution

A *connected* domain $\mathcal{D}_c$ of $\mathbb{R}^2$ is a set in which every two points in $\mathcal{D}_c$ can be connected by a curve lying entirely within $\mathcal{D}_c$. A set $\mathcal{D} \subseteq \mathbb{R}^2$ is *simply connected* if it is connected, and if any curve between two points can be continuously contracted, staying within $\mathcal{D}$, into another curve with the same endpoints.

The *divergence* of function $F(x) = (F_1(x_1, x_2), F_2(x_1, x_2))$ is defined as the quantity

$$\text{div}F(x) = \frac{\partial F_1}{\partial x_1}(x_1, x_2) + \frac{\partial F_2}{\partial x_2}(x_1, x_2). \tag{41}$$

Theorem 4.1 (Bendixson's Theorem)**.** *Let $\mathcal{D}$ be a simply connected of $\mathbb{R}^2$ such that the divergence of F, given by (41), is not identically zero over any subregion of $\mathcal{D}$, and does not change sign in $\mathcal{D}$. Then $\mathcal{D}$ does not contain any cycle of the system.*

4.2. Existence of a periodic solution

Theorem 4.2 (Poincaré-Bendixson's Theorem). *Let $x(t)$ be a solution, and let $\mathcal{S}_\omega$ (respectively, $\mathcal{S}_\alpha$) denote its ω-limit set (resp., α-limit set). If $\mathcal{S}_\omega$ (respectively, $\mathcal{S}_\alpha$) is contained in a compact region $\mathcal{M} \subset \mathbb{R}^2$, and if $\mathcal{M}$ does not contain any equilibrium point of the system, then $\mathcal{S}_\omega$ (respectively, $\mathcal{S}_\alpha$) is a cycle of the system.*

5. Bifurcations

To make parameters explicitly appear in the state equations, we recast them as

$$\dot{x}(t) = F(x(t), \mu) \tag{42}$$

in continuous-time, and by

$$x(t+1) = F(x(t), \mu) \tag{43}$$

in discrete time. Function $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is at least a C^1-function, i.e. a continuously differentiable function.

Definition 5.1. *The system (42) or (43) undergoes a bifurcation at μ_0, if there is no neighborhood $\mathcal{V}$ of μ_0 on the real line $\mathbb{R}$ such that all systems with $\mu \in \mathcal{V}$ have the same qualitative behavior.*

5.1. Implicit Function Theorem

Theorem 5.1 (Implicit Function Theorem). *Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be a C^1-function and suppose that*

$$F(x_0, y_0) = 0 \tag{44}$$

with $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$. Suppose that the $n \times n$ Jacobian matrix of F with respect to x is

$$J_x(x_0, y_0) = \frac{\partial F}{\partial x}(x_0, y_0) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(x_0, y_0) & \frac{\partial F_1}{\partial x_2}(x_0, y_0) & \dots & \frac{\partial F_1}{\partial x_n}(x_0, y_0) \\ \frac{\partial F_2}{\partial x_1}(x_0, y_0) & \frac{\partial F_2}{\partial x_2}(x_0, y_0) & \dots & \frac{\partial F_2}{\partial x_n}(x_0, y_0)) \\ \vdots & & \ddots & \\ \frac{\partial F_n}{\partial x_1}(x_0, y_0) & \frac{\partial F_n}{\partial x_2}(x_0, y_0) & \dots & \frac{\partial F_n}{\partial x_n}(x_0, y_0) \end{bmatrix} \tag{45}$$

is non-singular (i.e is invertible). Then there is a neighborhood $\mathcal{U}$ of (x_0, y_0) in $\mathbb{R}^{n+m}$, a neighborhood $\mathcal{V}$ of y_0 in $\mathbb{R}^m$ and a C^1- function $g : \mathcal{V} \to \mathbb{R}^n$ such that all solutions of $F(x, y) = 0$ in $\mathcal{U}$ are given by $x = g(y)$. Moreover,

$$\begin{aligned} \frac{\partial g}{\partial y}(y_0) &= -\left(\frac{\partial F}{\partial x}\right)^{-1}(x_0, y_0) \cdot \frac{\partial F}{\partial y}(x_0, y_0) \\ &= -J_x^{-1}(x_0, y_0) J_y(x_0, y_0). \end{aligned} \tag{46}$$

5.2. Fold bifurcation of equilibrium/fixed points in 1-dim. state-space

Theorem 5.2 (Fold Bifurcation in 1-dim systems). *Let the continuous-time (respectively, discrete-time) system given by*

$$\dot{x}(t) = F(x(t), \mu),$$

respectively by

$$x(t+1) = F(x(t), \mu),$$

and $F : \mathbb{R}^2 \to \mathbb{R}$ *be* C^2*-function (twice continuously differentiable). Let* $\overline{x}_0 \in \mathbb{R}$ *and* $\mu_0 \in \mathbb{R}$ *be such that*

$$\begin{aligned}
F(\overline{x}_0, \mu_0) &= 0 \qquad (\text{ resp.}, \ = \overline{x}_0) \\
\frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) &= 0 \qquad (\text{ resp.}, \ = 1) \\
\frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0) &\neq 0 \\
\frac{\partial F}{\partial \mu}(\overline{x}_0, \mu_0) &\neq 0.
\end{aligned}$$

Then the system undergoes a fold bifurcation at $(\overline{x}_0, \mu_0)$*. That is, in a neighborhood of* $(\overline{x}_0, \mu_0)$*:*

(i) for $\mu < \mu_0$*, there are two equilibrium/fixed points, one asymptotically stable, the other unstable, and for* $\mu > \mu_0$ *there is none, or vice-versa;*

(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

$$\dot{x}(t) = \mu \pm x^2(t) \tag{47}$$

for a continuous-time system, or

$$x(t+1) = \mu + x(t) \pm x^2(t). \tag{48}$$

for a discrete-time system.

5.3. Transcritical bifurcation of equilibrium/fixed points in 1-dim. state-space

Theorem 5.3 (Transcritical Bifurcation in 1-dim systems)**.** *Let the continuous-time (respectively, discrete-time) system given by*

$$\dot{x}(t) = F(x(t), \mu),$$

respectively by

$$x(t+1) = F(x(t), \mu),$$

and let $F : \mathbb{R}^2 \to \mathbb{R}$ *be a* C^2*-function (two times continuously differentiable).* $F : \mathbb{R}^2 \to \mathbb{R}$ *be* C^2*-function (twice continuously differentiable). Let* $\overline{x}_0 \in \mathbb{R}$ *and* $\mu_0 \in \mathbb{R}$ *be such that*

$$\begin{aligned}
F(\overline{x}_0, \mu_0) &= 0 \qquad (\text{ resp.}, \ = \overline{x}_0) \\
\frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) &= 0 \qquad (\text{ resp.}, \ = 1) \\
\frac{\partial F}{\partial \mu}(\overline{x}_0, \mu_0) &= 0 \\
\frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0) &\neq 0 \\
\left[\frac{\partial^2 F}{\partial \mu \partial x}(\overline{x}_0, \mu_0)\right]^2 - \frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0)\frac{\partial^2 F}{\partial \mu^2}(\overline{x}_0, \mu_0) &> 0.
\end{aligned}$$

Then the system undergoes a transcritical (also called saddle-node) bifurcation at $(\overline{x}_0, \mu_0)$*. That is, in a neighborhood of* $(\overline{x}_0, \mu_0)$*:*

(i) for $\mu \neq \mu_0$*, there are two equilibrium/fixed points, one asymptotically stable, the other unstable. They switch stability at* $\mu = \mu_0$*;*

(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

$$\dot{x}(t) = \mu x(t) \pm x^2(t) \tag{49}$$

for a continuous-time system, or

$$x(t+1) = (1+\mu)x(t) \pm x^2(t). \tag{50}$$

for a discrete-time system.

5.4. Pitchfork bifurcation of equilibrium/fixed points in 1-dim. state-space

Theorem 5.4 (Pitchfork Bifurcation in 1-dim systems)**.** *Let the continuous-time (respectively, discrete-time) system given by*

$$\dot{x}(t) = F(x(t), \mu),$$

respectively by

$$x(t+1) = F(x(t), \mu),$$

and let $F : \mathbb{R}^2 \to \mathbb{R}$ *be a* C^3*-function (three times continuously differentiable), which is odd (i.e. verifies* $F(-x, \mu) = -F(x, \mu)$ *for all* $x \in \mathbb{R}$ *and* $\mu \in \mathbb{R}$*). Let* $\mu_0 \in \mathbb{R}$ *be such that*

$$\begin{aligned} \frac{\partial F}{\partial x}(0, \mu_0) &= 0 \qquad (\textit{ resp.}, \ = 1) \\ \frac{\partial^2 F}{\partial x \partial \mu}(0, \mu_0) &\neq 0 \\ \frac{\partial^3 F}{\partial x^3}(0, \mu_0) &\neq 0. \end{aligned}$$

Then the system undergoes a pitchfork bifurcation at $(0, \mu_0)$*, that is, in a neighborhood of* $(\overline{x}_0, \mu_0)$*,*

(i) for $\mu < \mu_0$*, the origin is the only equilibrium/fixed point and it is asymptotically stable, whereas for* $\mu > \mu_0$ *the origin is an unstable equilibrium/fixed point, and in addition, there are two asymptotically stable equilibrium/fixed points, or vice-versa (this is called a* supercritical pitchfork bifurcation*) or for* $\mu < \mu_0$*, the origin is an asymptotically stable equilibrium/fixed point and in addition there are two unstable equilibrium/fixed points, whereas for* $\mu > \mu_0$ *the origin is the only equilibrium/fixed point and it is unstable, or vice-versa (this is called a* subcritical pitchfork bifurcation*);*

(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

$$\dot{x}(t) = \mu x(t) \pm x^3(t) \tag{51}$$

for a continuous-time system, or

$$x(t+1) = (1+\mu)x(t) \pm x^3(t) \tag{52}$$

for a discrete-time system.

5.5. Flip bifurcation of fixed points in 1-dim. state-space

Theorem 5.5 (Flip Bifurcation in 1-dim systems). *Let the discrete-time system*

$$x(t+1) = F(x(t), \mu),$$

given by the C^3-function $F : \mathbb{R}^2 \to \mathbb{R}$. Let $\overline{x}_0 \in \mathbb{R}$ and $\mu_0 \in \mathbb{R}$ be such that

$$\begin{aligned} F(\overline{x}_0, \mu_0) &= \overline{x}_0 \\ \frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) &= -1 \\ \left[\frac{\partial^2 F}{\partial \mu \partial x} + \frac{1}{2}\left(\frac{\partial F}{\partial \mu}\right)\left(\frac{\partial^2 F}{\partial x^2}\right)\right](\overline{x}_0, \mu_0) = \alpha &\neq 0 \\ \frac{1}{6}\frac{\partial^3 F}{\partial x^3}(\overline{x}_0, \mu_0) + \left(\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0)\right)^2 = \beta &\neq 0. \end{aligned}$$

Then the system undergoes a flip bifurcation at $(\overline{x}_0, \mu_0)$, that is, in a neighborhood of $(\overline{x}_0, \mu_0)$,

(i) for $\mu < \mu_0$, there is an asymptotically stable fixed point, whereas for $\mu > \mu_0$ the fixed point is unstable, and in addition, there is an asymptotically stable 2-cycle, or vice-versa (this is called a supercritical flip bifurcation*) or for $\mu < \mu_0$, there is an asymptotically stable fixed point and an unstable 2-cycle, whereas for $\mu > \mu_0$ there is only the fixed point and it is unstable, or vice-versa (this is called a* subcritical flip bifurcation*);*

(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

$$x(t+1) = -(1+\mu)x(t) \pm x^3(t). \tag{53}$$

5.6. Andronov-Hopf bifurcation of equilibrium points in 2-dim. state-space

Theorem 5.6 (Andronov-Hopf Bifurcation in 2-dim systems). *Let the continuous-time system given by*

$$\dot{x}(t) = F(x(t), \mu),$$

and let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be a C^4-function. Let $\overline{x}_0 \in \mathbb{R}^2$ and $\mu_0 \in \mathbb{R}$ be such that $F(\overline{x}_0, \mu_0) = (0, 0)$ and that the Jacobian matrix $\partial F/\partial x(\overline{x}_0, \mu_0)$ has imaginary eigenvalues $\lambda_0 = j\omega_0$ and $\lambda_0^* = -j\omega_0$.

If

$$\frac{d\Re(\lambda(\mu))}{d\mu}(\mu_0) \neq 0 \tag{54}$$

and a complicated non-degeneracy condition is met, which we will not specify here (and which therefore can always be assumed to be satisfied in the exercises and exams), then the system undergoes an Andronov-Hopf bifurcation at $(\overline{x}_0, \mu_0)$, that is, in a neighborhood of $(\overline{x}_0, \mu_0)$,

(i) for $\mu < \mu_0$, there is an asymptotically stable equilibrium point $\overline{x}(\mu)$, whereas for $\mu > \mu_0$ the equilibrium point $\overline{x}(\mu)$ becomes unstable, and in addition, there is a stable periodic solution, or vice-versa (this is called a supercritical Andronov-Hopf bifurcation*) or for $\mu < \mu_0$, there is an asymptotically stable equilibrium point $\overline{x}(\mu)$ and an unstable periodic solution, whereas for $\mu > \mu_0$ there is only the equilibrium point $\overline{x}(\mu)$ and it is unstable, or vice-versa (this is called a* subcritical Andronov-Hopf bifurcation*);*

(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

$$\begin{aligned}\dot{x}_1 &= \mu x_1 - x_2 \pm x_1 \left(x_1^2 + x_2^2\right) \\ \dot{x}_2 &= x_1 - \mu x_2 \pm x_2 \left(x_1^2 + x_2^2\right).\end{aligned}$$

(iii) the period of the periodic solution is a differentiable function $T(\mu)$ of μ, with $T(\mu_0) = 2\pi/\omega_0$.

6. Chaos in 1-dim discrete-time systems

6.1. Lyapunov Exponents for 1-dim. Maps

We discuss here only 1-dimensional discrete-time systems

$$x(t+1) = F(x(t)) \tag{55}$$

where $F : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

Definition 6.1. *The* Lyapunov exponent *of a solution $x(t)$ of the autonomous 1-dim discrete-time system (55) is given by*

$$\alpha = \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \ln \left| \frac{dF}{dx}(x(\tau)) \right| \tag{56}$$

if the limit exists.

Theorem 6.1. *If P is an invariant measure under $F(\cdot)$, then for P-almost all solutions $x(t)$, the Lyapunov exponent (56) exists. If, in addition, P is ergodic with respect to $F(\cdot)$, then for P-almost all solutions, the Lyapunov exponent (56) is the same and its value is given by*

$$\alpha = \int_{-\infty}^{\infty} \ln \left| \frac{dF}{dx}(x) \right| dP(x). \tag{57}$$

If the ergodic invariant measure is given by a density $\rho(x)$, i.e. $dP(x) = \rho(x)dx$, then (57) becomes

$$\alpha = \int_{-\infty}^{\infty} \ln \left| \frac{dF}{dx}(x) \right| \rho(x)dx. \tag{58}$$

A. Useful facts from Linear Algebra and Calculus

A.1. Determinant, Inverse, Taylor Expansion, Polar Coordinates

- The determinant of a 3-by-3 matrix is given by:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg. \tag{59}$$

- The inverse of a 2-by-2 matrix is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{60}$$

  The inverse of a 3-by-3 matrix is:

$$A^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\det A} \begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}, \tag{61}$$

  where:
  $A = (ei - fh)$ $D = (ch - bi)$ $G = (bf - ce)$
  $B = (fg - di)$ $E = (ai - cg)$ $H = (cd - af)$
  $C = (dh - eg)$ $F = (bg - ah)$ $K = (ae - bd)$.

- Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and λ_1, λ_2 denote the eigenvalues of A. We have:

$$Tr(A) = a + d = \lambda_1 + \lambda_2 \tag{62}$$

$$\det(A) = \lambda_1 \lambda_2 \tag{63}$$

- **Taylor expansion of a function of one variable**
  Expansion of the function $f(x)$ in a neighborhood of a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n. \tag{64}$$

- **Taylor expansion of a function of two variables**
  Second order Tayler serie of the function $f(x, y)$ around the point (a, b):

$$\begin{aligned} f(x, y) \approx f(a, b) + (x - a)\frac{\partial f}{\partial x}(a, b) + (y - b)\frac{\partial f}{\partial y}(a, b) \\ + \frac{1}{2!}\left[(x - a)^2 \frac{\partial^2 f}{\partial x^2}(a, b) + 2(x - a)(y - b)\frac{\partial^2 f}{\partial x \partial y}(a, b) + (y - b)^2 \frac{\partial^2 f}{\partial y^2}(a, b)\right]. \end{aligned}$$

- **Polar coordinates representation:**

$$r = \sqrt{x_1^2 + x_2^2}, \quad \varphi = \arctan \frac{x_2}{x_1} \quad \Leftrightarrow \quad x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi \tag{65}$$

A.2. Jordan Factorization

Any $n \times n$ matrix with $1 \leq q \leq n$ linearly independent eigenvectors is similar to a matrix J in *Jordan (canonical) form* with q square blocks on its diagonal, which reads

$$J = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_q \end{bmatrix}.$$

This means that there exists a $n \times n$ invertible matrix S such that

$$A = SJS^{-1}.$$

Each Jordan block J_i corresponds to one eigenvalue λ_i and to one (unit-norm) eigenvector, and has the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix}.$$

In other words, the elements on the main diagonal are all equal to λ_i, the elements on the first upper diagonal are all equal to 1 and all the other entries of the Jordan block are 0.

The *geometric multiplicity* of an eigenvalue λ_k is equal to the dimension of its eigenspace, i.e. the number of linearly independent eigenvectors with eigenvalue λ_i, and thus to the number of Jordan blocks corresponding to this eigenvalue.

The *algebraic multiplicity* of an eigenvalue λ_i is the number of times it is a root of the characteristic polynomial of A, i.e. the number of times it appears on the diagonal of J. It is equal to the sum of the dimensions of all Jordan blocks corresponding to the eigenvalue λ_i.

The canonical Jordan form of A allows to express the tth power of A as

$$A^t = SJ^tS^{-1} \tag{66}$$

where

$$J^t = \begin{bmatrix} J_1^t & 0 & \dots & 0 \\ 0 & J_2^t & \ddots & \vdots \\ \vdots & \ddots & J_{q-1}^t & 0 \\ 0 & \dots & 0 & J_q^t \end{bmatrix},$$

and where one can compute that

$$J_i^t = \begin{bmatrix} \lambda_i^t & t\lambda_i^{t-1} & \frac{t(t-1)}{2}\lambda_i^{t-2} & \dots & \binom{t}{n_i}\lambda_i^{t-n_i} \\ 0 & \lambda_i^t & t\lambda_i^{t-1} & \frac{t(t-1)}{2}\lambda_i^{t-2} & \dots \\ \vdots & & \ddots & \ddots & \\ & & & & \\ 0 & & 0 & \lambda_i^t & t\lambda_i^{t-1} \\ 0 & \dots & & 0 & \lambda_i^t \end{bmatrix}.$$

The matrix $S = [S_1, S_2, \dots, S_q]$ has q (rectangular $n \times n_i$) blocks $S_i \in \mathbb{C}^{n \times n_i}$, each of which contains the columns of S associated with the Jordan block J_i. Let $S_i = [v_{i,1}\ v_{i,2}\ \dots\ v_{i,n_i}]$. We can find the columns of S_i iteratively in the following way:

$$\begin{aligned} (A - \lambda_i I_n)v_{i,1} &= 0 \\ (A - \lambda_i I_n)v_{i,j} &= v_{i,(j-1)}. \end{aligned}$$

The first column $v_{i,1}$ is the eigenvector of A associated to the eigenvalue λ_i. The other $n_i - 1$ columns are called the generalized eigenvectors associated to the eigenvalue λ_i.

A.3. Laplace and z-Transforms

Table 1: Some Laplace transforms

$f(t)$	$F(s)$
$\delta(t)$	1
1	$1/s$
t	$1/s^2$
$t^{n-1}/(n-1)!$	$1/s^n$
e^{at}	$1/(s-a)$
$t^{n-1}e^{at}/(n-1)!$	$1/(s-a)^n$
$\cos(at)$	$s/(s^2+a^2)$
$\sin(at)$	$a/(s^2+a^2)$
$\cos(at)e^{bt}$	$\frac{s-b}{(s-b)^2+a^2}$
$\sin(at)e^{bt}$	$\frac{a}{(s-b)^2+a^2}$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f(at)$	$F(s/a)/a$
$f(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$\frac{df}{dt}(t)$	$sF(s)-f(0)$
$\int_0^t f(\tau)d\tau$	$F(s)/s$
$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$

Table 2: Some (unilateral) z-transforms

$f(t)$	$F(z)$
$\delta(t)$	1
1	$1/(1-z^{-1})$
t	$z^{-1}/(1-z^{-1})^2$
a^t	$1/(1-az^{-1})$
ta^t	$az^{-1}/(1-az^{-1})^2$
$\cos(at)$	$\frac{1-z^{-1}\cos(a)}{1-2z^{-1}\cos(a)+z^{-2}}$
$\sin(at)$	$\frac{1-z^{-1}\sin(a)}{1-2z^{-1}\cos(a)+z^{-2}}$
$\cos(at)b^t$	$\frac{1-bz^{-1}\cos(a)}{1-2bz^{-1}\cos(a)+b^2z^{-2}}$
$\sin(at)b^t$	$\frac{1-bz^{-1}\sin(a)}{1-2bz^{-1}\cos(a)+b^2z^{-2}}$
$af(t)+bg(t)$	$aF(z)+bG(z)$
$f(t-a)$	$z^{-a}F(z)$
$a^tf(t)$	$F(z/a)$
$(f\star g)(t)$	$F(z)G(z)$