INDY

Bifurcations

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Definition

- Qualitative behavior of dynamical systems depend on parameters. Here we consider 1 parameter $\mu \in \mathbb{R}$
- Make parameter dependence explicit :
  - $\dot{x} = F(x) \longrightarrow \dot{x} = F(x, \mu)$
  - $x(t+1) = F(x(t)) \longrightarrow x(t+1) = F(x(t), \mu)$
  - $F: \mathbb{R}^{n+1} \to \mathbb{R}^n$ continuously differentiable (at least C^1)
- The system undergoes a bifurcation at μ_0 if there is no neighborhood $\mathcal{V} \subset \mathbb{R}$ of μ_0 such that all systems with $\mu \in \mathcal{V}$ have the same qualitative behavior.
- Same qualitative behavior ≡ there is a continuous coordinate and time transformation mapping the solutions of one system to the solutions of the other, and vice versa.
- Codimension of a bifurcation = number of parameters that must be varied for the bifurcation to occur.
  - Here we consider only bifurcations of codimension 1 ($\mu \in \mathbb{R}$)

Implicit Function Theorem

Theorem 7.1 (Implicit Function Theorem). *Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be a C^1-function and suppose that*

$$F(x_0, y_0) = 0 \tag{7.6}$$

with $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$. Suppose that the $n \times n$ Jacobian matrix of F with respect to x is

$$J_x(x_0, y_0) = \frac{\partial F}{\partial x}(x_0, y_0) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(x_0, y_0) & \frac{\partial F_1}{\partial x_2}(x_0, y_0) & \cdots & \frac{\partial F_1}{\partial x_n}(x_0, y_0) \\ \frac{\partial F_2}{\partial x_1}(x_0, y_0) & \frac{\partial F_2}{\partial x_2}(x_0, y_0) & \cdots & \frac{\partial F_2}{\partial x_n}(x_0, y_0)) \\ \vdots & & \ddots & \\ \frac{\partial F_n}{\partial x_1}(x_0, y_0) & \frac{\partial F_n}{\partial x_2}(x_0, y_0) & \cdots & \frac{\partial F_n}{\partial x_n}(x_0, y_0) \end{bmatrix} \tag{7.7}$$

is non-singular (i.e is invertible). Then there is a neighborhood $\mathcal{U}$ of (x_0, y_0) in $\mathbb{R}^{n+m}$, a neighborhood $\mathcal{V}$ of y_0 in $\mathbb{R}^m$ and a C^1- function $g : \mathcal{V} \to \mathbb{R}^n$ such that all solutions of $F(x, y) = 0$ in $\mathcal{U}$ are given by $x = g(y)$. Moreover,

$$\begin{aligned} \frac{\partial g}{\partial y}(y_0) &= -\left(\frac{\partial F}{\partial x}\right)^{-1}(x_0, y_0) \cdot \frac{\partial F}{\partial y}(x_0, y_0) \\ &= -J_x^{-1}(x_0, y_0) J_y(x_0, y_0). \end{aligned} \tag{7.8}$$

Application to Continuous Time System

- $\dot{x} = F(x, \mu)$ with $F: \mathbb{R}^{n+1} \to \mathbb{R}^n$ continuously differentiable (C^1)
- Any equilibrium point $\bar{x}$ satisfies $F(\bar{x}, \mu) = 0$
- Let $(\bar{x}_0, \mu_0)$ be such that $F(\bar{x}_0, \mu_0) = 0$
- When can we write $\bar{x} = g(\mu)$ in a neighborhood $\mathcal{V}$ of μ_0 with g a C^1 function?
- Implicit function Theorem:

If $J_x(\bar{x}_0, \mu_0) = \frac{\partial F}{\partial x}(\bar{x}_0, \mu_0)$ is non singular (i.e., all its eigenvalues are non-zero), then

- $\exists$ neighborhood $\mathcal{U} \subset \mathbb{R}^2$ of $(\bar{x}_0, \mu_0)$
- $\exists$ neighborhood $\mathcal{V} \subset \mathbb{R}$ of μ_0
- $\exists$ (unique) C^1 function $g: \mathcal{V} \to \mathbb{R}$ with $\bar{x}_0 = g(\mu_0)$ and such that

$$F(\bar{x}, \mu) = 0 \text{ for } (\bar{x}, \mu) \in \mathcal{U} \Leftrightarrow \bar{x} = g(\mu) \text{ for } \mu \in \mathcal{V}$$

- Moreover, $\frac{\partial g}{\partial \mu}(\mu_0) = -\left(\frac{\partial F}{\partial x}\right)^{-1}(\bar{x}_0, \mu_0) \cdot \frac{\partial F}{\partial \mu}(\bar{x}_0, \mu_0)$.

Application to Discrete Time System

- $x(t+1) = F(x(t), \mu)$ with $F: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ cont. different. (C^1)
- Any fixed point $\bar{x}$ satisfies $F(\bar{x}, \mu) - \bar{x} = 0$
- Let $(\bar{x}_0, \mu_0)$ be such that $F(\bar{x}_0, \mu_0) - \bar{x}_0 = 0$
- When can we write $\bar{x} = g(\mu)$ in a neighborhood $\mathcal{V}$ of μ_0 with g a C^1 function?
- Implicit function Theorem:

If $J_x(\bar{x}_0, \mu_0) - I_n = \frac{\partial F}{\partial x}(\bar{x}_0, \mu_0) - I_n$ is non singular (i.e., all the eigenvalues of $J_x(\bar{x}_0, \mu_0)$ are not equal to 1), then

- ∃ neighborhood $\mathcal{U} \subset \mathbb{R}^2$ of $(\bar{x}_0, \mu_0)$
- ∃ neighborhood $\mathcal{V} \subset \mathbb{R}$ of μ_0
- ∃ (unique) C^1 function $g: \mathcal{V} \rightarrow \mathbb{R}$ with $\bar{x}_0 = g(\mu_0)$ and such that

$$F(\bar{x}, \mu) - \bar{x} = 0 \text{ for } (\bar{x}, \mu) \in \mathcal{U} \Leftrightarrow \bar{x} = g(\mu) \text{ for } \mu \in \mathcal{V}$$

- Moreover, $\frac{\partial g}{\partial \mu}(\mu_0) = -\left(\frac{\partial F}{\partial x} - I_n\right)^{-1}(\bar{x}_0, \mu_0) \cdot \frac{\partial F}{\partial \mu}(\bar{x}_0, \mu_0)$.

Necessary Condition for Bifurcation

- Implicit equation:
  - (CT) Equilibrium point equation: $F(\bar{x}, \mu) = 0$.
  - (DT) Fixed point equation $F(\bar{x}, \mu) - \bar{x} = 0$.
- If Jacobian matrix (∂F/∂x) does not have
  - (CT) the eigenvalue 0
  - (DT) the eigenvalue 1
  - then in a neighborhood of $(\bar{x}_0, \mu_0)$, the equilibrium/fixed points are given by a continuously differentiable 1-parameter family $\bar{x}(\mu)$ with

$$\overline{x}(\mu_0) = \overline{x}_0$$

and

$$\frac{\partial \overline{x}}{\partial \mu}(\mu_0) = -\left(\frac{\partial F}{\partial x}\right)^{-1}(\overline{x}_0, \mu_0) \cdot \frac{\partial F}{\partial \mu}(\overline{x}_0, \mu_0).$$

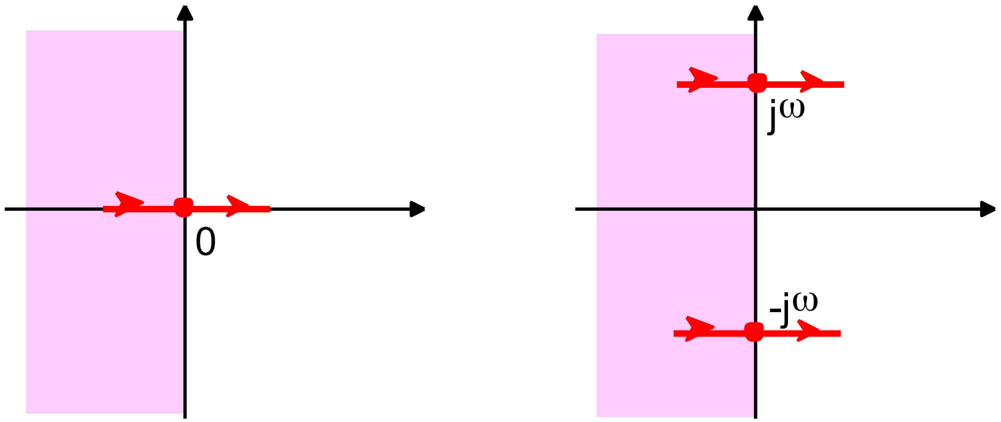
- A local bifurcation might however still occurs, if the local stability of the equilibrium/fixed point changes at $\mu = \mu_0$.
- If Jacobian matrix (∂F/∂x) is hyperbolic, i.e., does not have
  - (CT) eigenvalues on the imaginary axis
  - (DT) eigenvalues on the unit circle

then there is no bifurcation at μ_0.

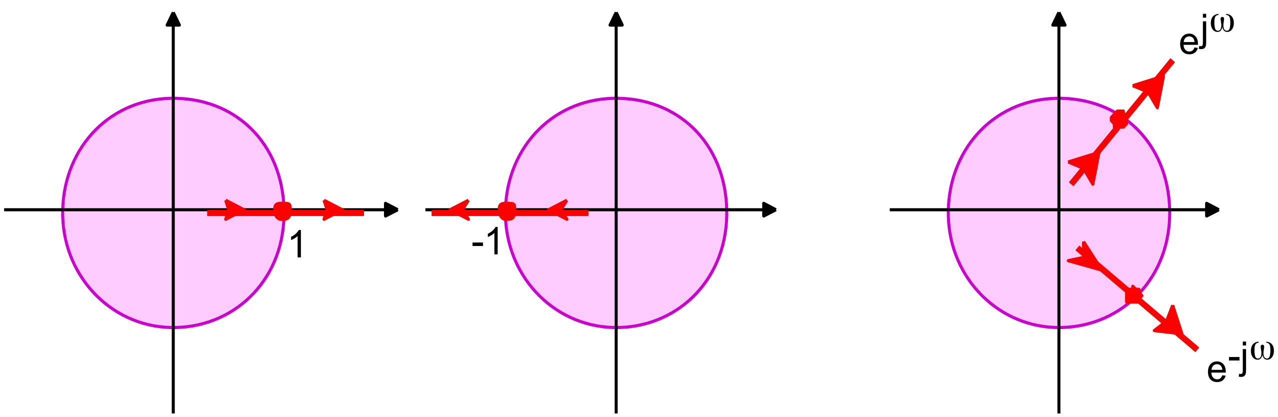
Simplest Bifurcations

- Bifurcations can only occur if Jacobian matrix ($\partial F/\partial x$) is non-hyperbolic

- Continuous-time system



- Discrete-time system



Theorem: Fold or Cusp Bifurcation

- $\dot{x} = F(x, \mu)$ with $(\bar{x}_0, \mu_0)$ such that

$$\begin{aligned} F(\overline{x}_0, \mu_0) &= 0 \\ \frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) &= 0 \\ \frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0) &\neq 0 \\ \frac{\partial F}{\partial \mu}(\overline{x}_0, \mu_0) &\neq 0. \end{aligned}$$

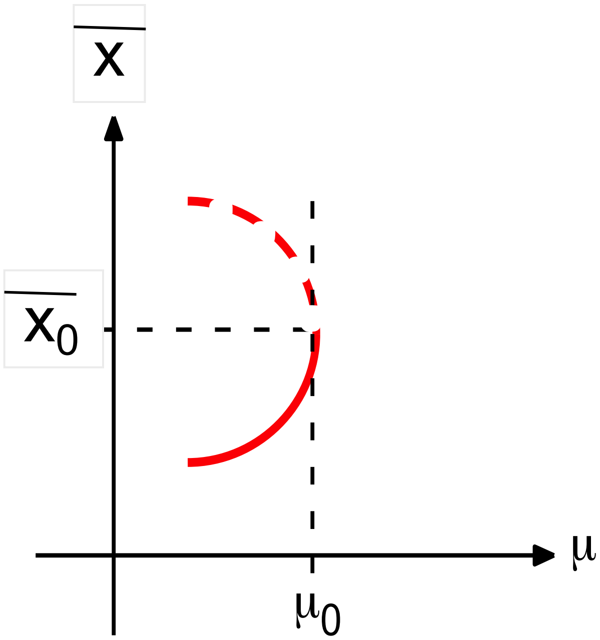
- Then the system undergoes a fold bifurcation $(\bar{x}_0, \mu_0)$, i.e. in a neighborhood of $(\bar{x}_0, \mu_0)$

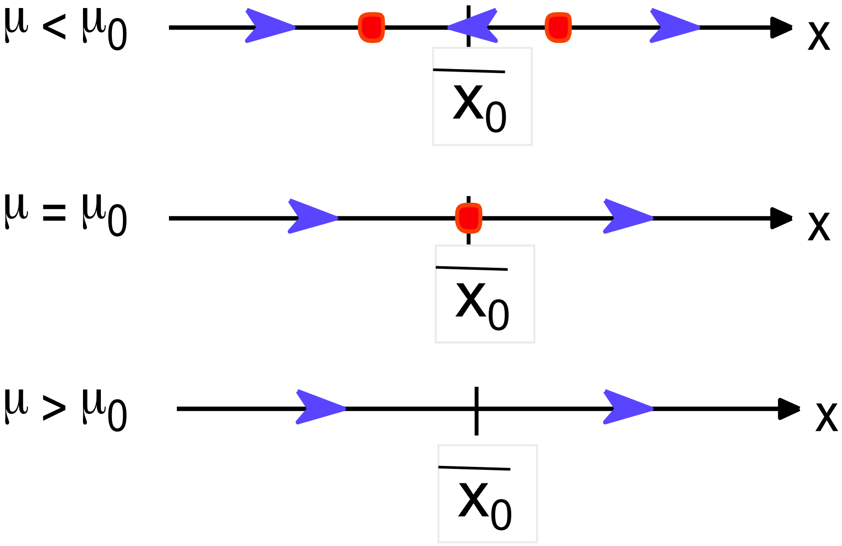
(i) for $\mu < \mu_0$, there are two equilibrium/fixed points, one asymptotically stable, the other unstable, and for $\mu > \mu_0$ there is none, or vice-versa;

(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

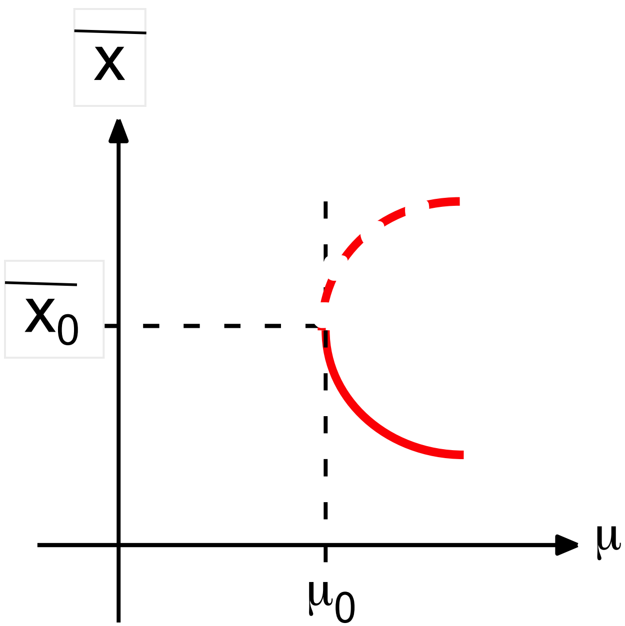
$$\dot{x}(t) = \mu \pm x^2(t) \tag{7.21}$$

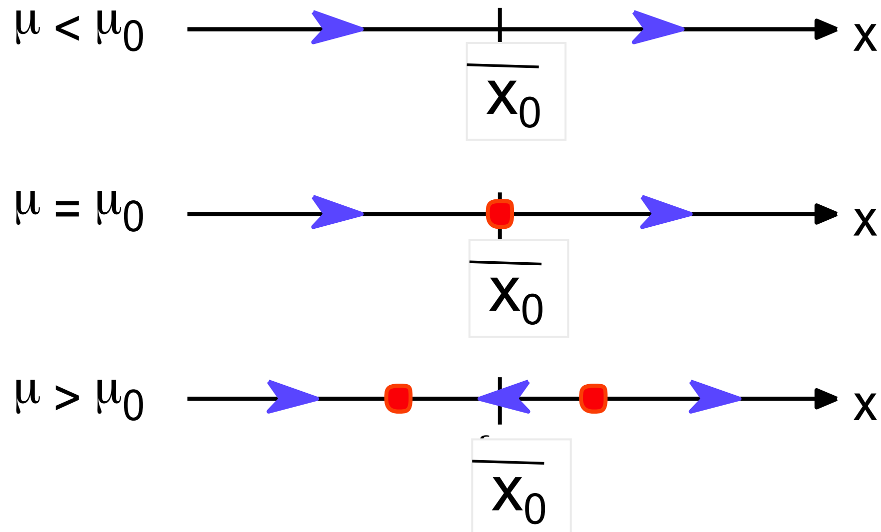
Fold Bifurcation: $a > 0$, $b > 0$



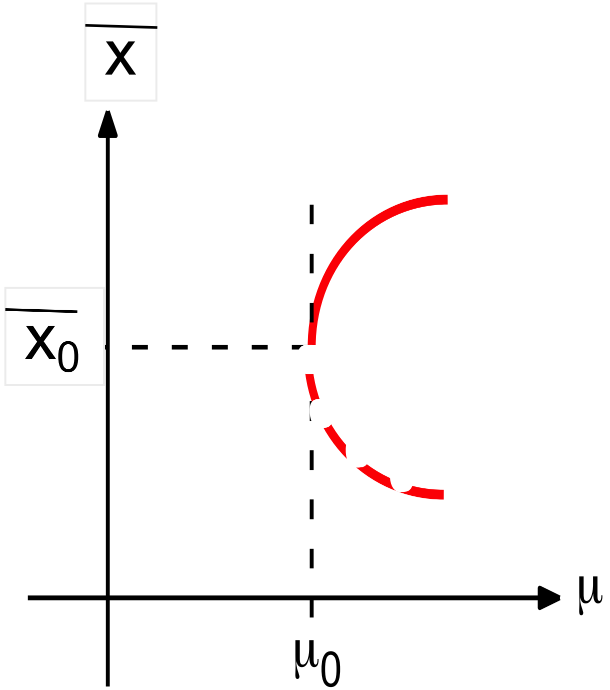


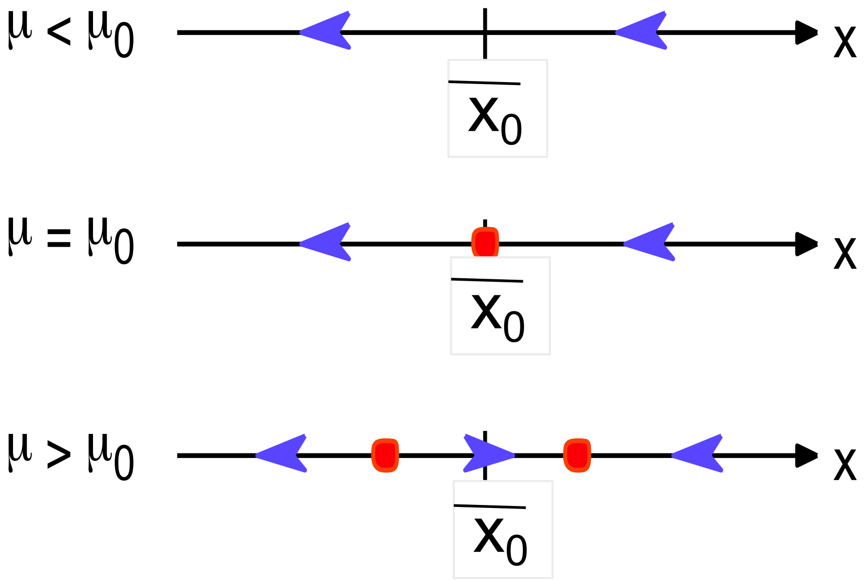
Fold Bifurcation: a > 0, b < 0



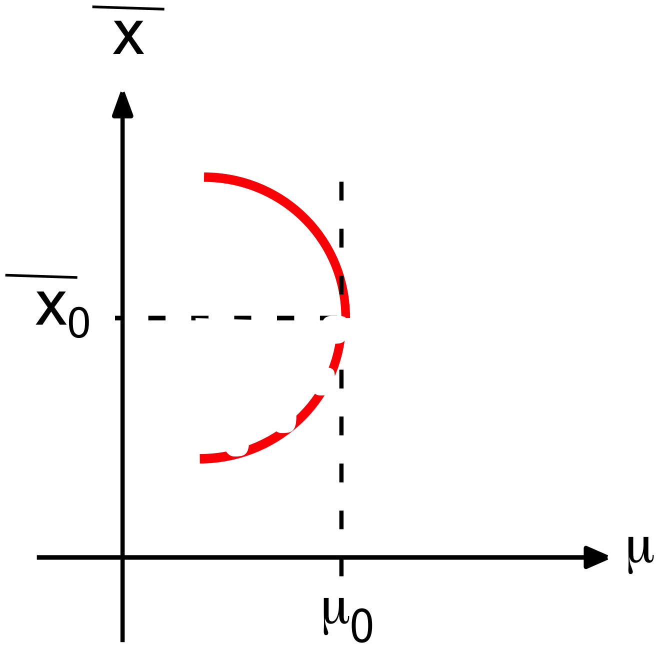


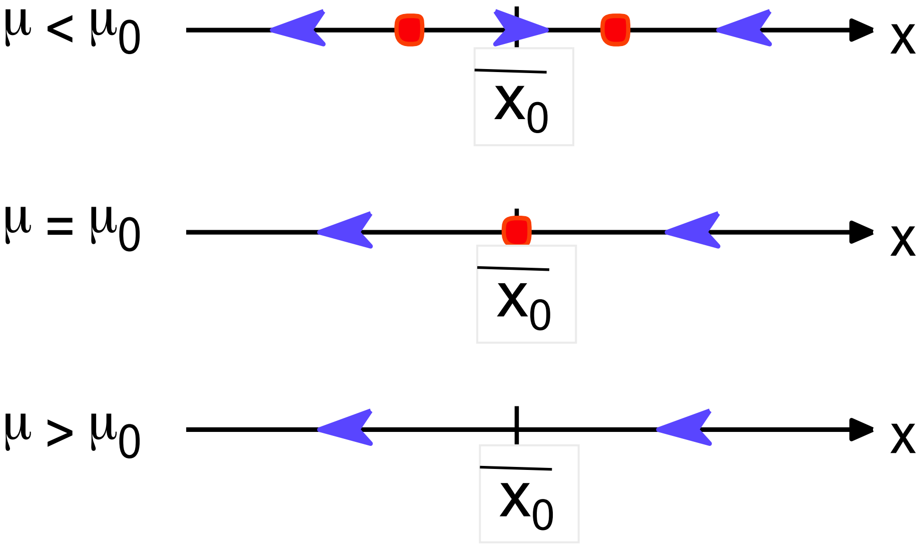
Fold Bifurcation: a < 0, b > 0





Fold Bifurcation: a < 0, b < 0





Theorem: Transcritical Bifurcation

- $\dot{x} = F(x, \mu)$ with $(\bar{x}_0, \mu_0)$ such that

$$\begin{aligned} F(\overline{x}_0, \mu_0) &= 0 \\ \frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) &= 0 \\ \frac{\partial F}{\partial \mu}(\overline{x}_0, \mu_0) &= 0 \\ \frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0) &\neq 0 \\ \left[\frac{\partial^2 F}{\partial \mu \partial x}\overline{x}_0, \mu_0)\right]^2 - \frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0)\frac{\partial^2 F}{\partial \mu^2}(\overline{x}_0, \mu_0) &> 0 \end{aligned}$$

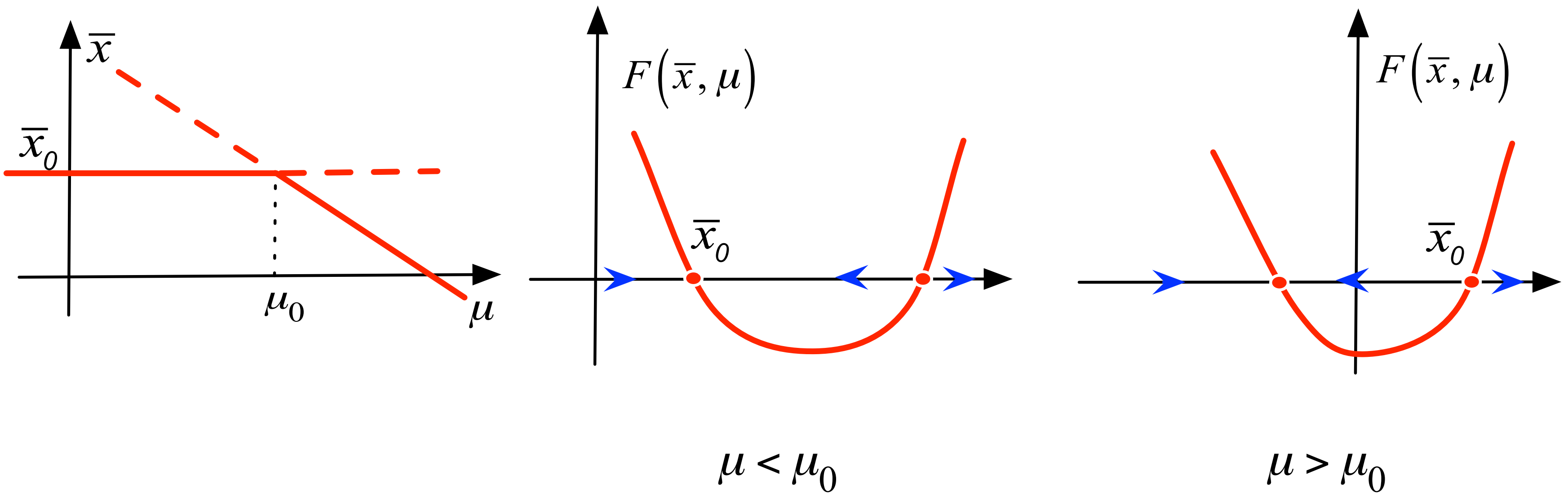
- Then the system undergoes a transcritical bifurcation at $(\bar{x}_0, \mu_0)$, i.e., in a neighborhood of $(\bar{x}_0, \mu_0)$

(i) for $\mu \neq \mu_0$, there are two equilibrium/fixed points, one asymptotically stable, the other unstable. They switch stability at $\mu = \mu_0$;

(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

$$\dot{x}(t) = \mu x(t) \pm x^2(t) \tag{7.37}$$

Transcritical Bifurcation: a > 0, b > 0



Example: SIS Epidemics

- $\frac{dI}{dt} = \beta S(t)I(t) + \gamma I(t)$ with $S(t) + I(t) = 1.$
- Let $R = {}^{\beta}/_{\gamma}$ and $\tau = \gamma t$. Then we can rewrite the state equation as

$\dot{I}(\tau) = \frac{dI}{d\tau}(\tau) = R(1 - I(\tau))I(\tau) + I(\tau) = -R\, I^2(\tau) + (R-1)I(\tau)$

- $F(I,R) = -R\, I^2 + (R-1)I \quad \Rightarrow \frac{\partial F}{\partial I}(I,R) = -2RI + R - 1$
- $F(I,R) = 0 \Rightarrow \bar{I}_h = 0$ or $\bar{I}_e = 1 - \frac{1}{R}$
- $\frac{\partial F}{\partial I}(I,R) = -2RI + R - 1 = 0 \ \Rightarrow \bar{I}_h = 0; R_0 = 1$ or $\bar{I}_e = 1 - \frac{1}{R_0}; R_0 = 1$
- $(\bar{I}_0, R_0) = (0,1) \Rightarrow F(\bar{I}_0, R_0) = 0$ and $\frac{\partial F}{\partial I}(\bar{I}_0, R_0) = 0$
- Other conditions:
  - $\frac{\partial F}{\partial \mu}(\bar{I}_0, R_0) = -\bar{I}_0^2 + \bar{I}_0 = 0$
  - ... $\neq 0$
  - ... $\neq 0.$
- The system undergoes a transcritical bifurcation at $(\bar{I}_0, R_0) = (0,1)$

Theorem: Pitchfork Bifurcation (CT)

- $\dot{x} = F(x, \mu)$ with $F(x, \mu) = -F(-x, \mu)$ and $(\bar{x}_0, \mu_0) = (0, \mu_0)$ such that

$$\begin{aligned} \frac{\partial F}{\partial x}(0, \mu_0) &= 0 \\ \frac{\partial^2 F}{\partial x \partial \mu}(0, \mu_0) &\neq 0 \\ \frac{\partial^3 F}{\partial x^3}(0, \mu_0) &\neq 0. \end{aligned}$$

- Then the system undergoes a pitchfork bifurcation at $(0, \mu_0)$, i.e., in a neighborhood of $(0, \mu_0)$

(i) for $\mu < \mu_0$, the origin is the only equilibrium/fixed point and it is asymptotically stable, whereas for $\mu > \mu_0$ the origin is an unstable equilibrium/fixed point, and in addition, there are two asymptotically stable equilibrium/fixed points, or vice-versa (this is called a supercritical pitchfork bifurcation*) or for $\mu < \mu_0$, the origin is an asymptotically stable equilibrium/fixed point and in addition there are two unstable equilibrium/fixed points, whereas for $\mu > \mu_0$ the origin is the only equilibrium/fixed point and it is unstable, or vice-versa (this is called a* subcritical pitchfork bifurcation*);*

(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

$$\dot{x}(t) = \mu x(t) \pm x^3(t) \tag{7.29}$$

Theorem: Pitchfork Bifurcation (DT)

- $x(t+1) = F(x(t), \mu)$ with $F(x, \mu) = -F(-x, \mu)$ and $(\bar{x}_0, \mu_0) = (0, \mu_0)$ such that

$$\begin{aligned} \frac{\partial F}{\partial x}(0, \mu_0) &= 1 \\ \frac{\partial^2 F}{\partial x \partial \mu}(0, \mu_0) &\neq 0 \\ \frac{\partial^3 F}{\partial x^3}(0, \mu_0) &\neq 0. \end{aligned}$$

- Then the system undergoes a pitchfork bifurcation at $(0, \mu_0)$, i.e., in a neighborhood of $(0, \mu_0)$

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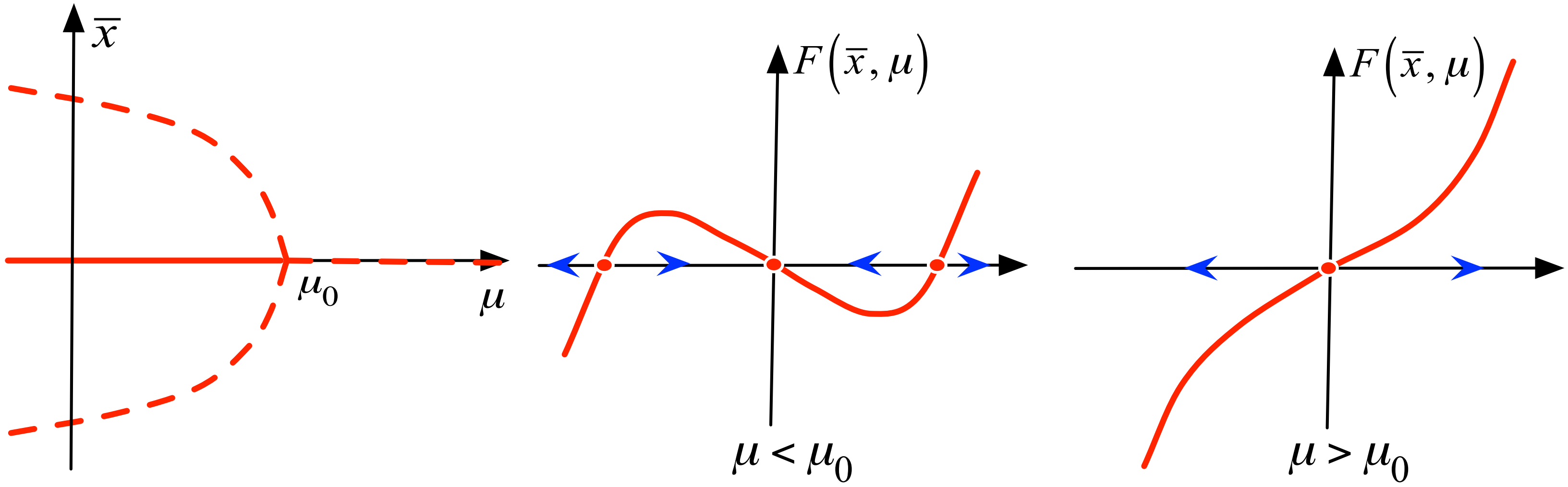
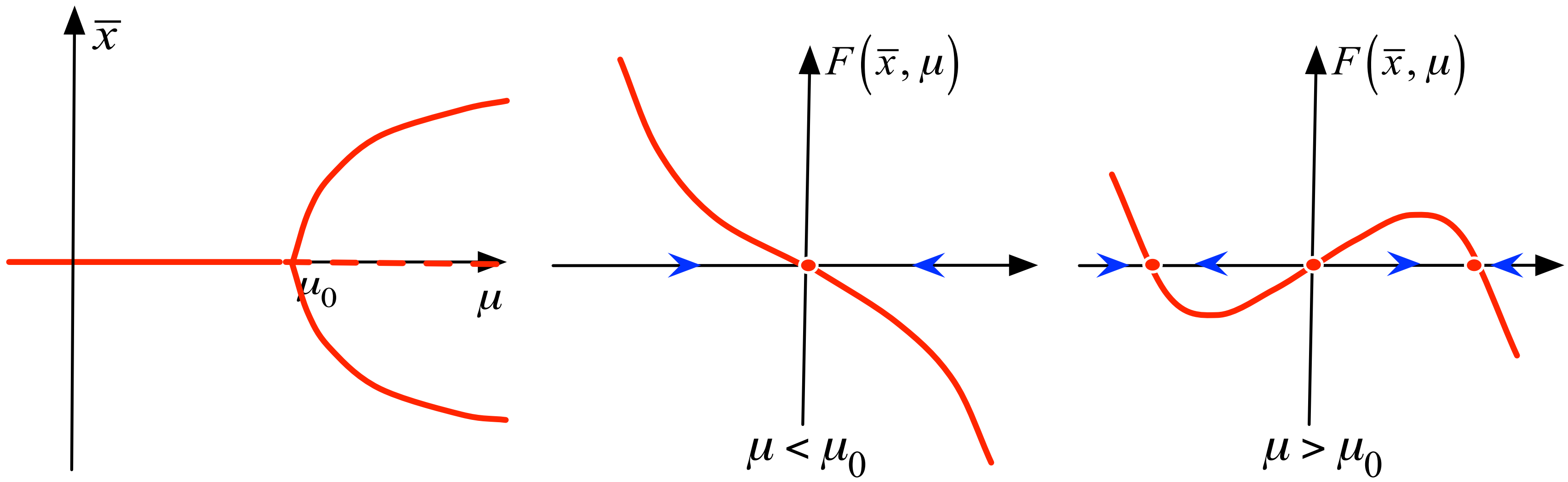
(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

$$x(t+1) = (1+\mu)x(t) \pm x^3(t) \tag{7.30}$$

Pitchfork Bifurcation: a > 0, b > 0 (subcritical)



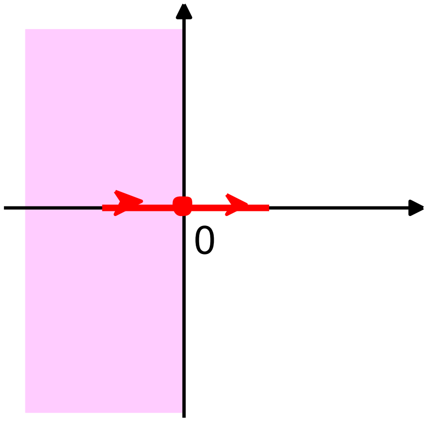
Pitchfork Bifurcation: a > 0, b < 0 (supercritical)



Simplest Bifurcations

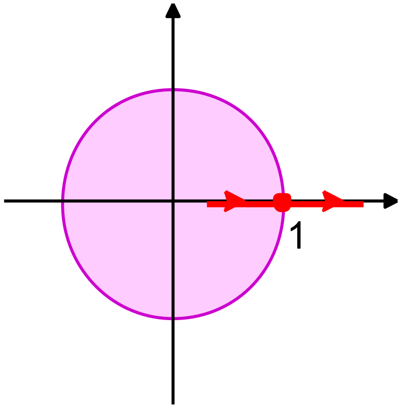
- ❑ Bifurcations can only occur if Jacobian matrix ($\partial F/\partial x$) is non-hyperbolic.

- ❑ Continuous-time system



- Fold (generic)
- Transcritical (symmetry)
- Pitchfork (symmetry)

- ❑ Discrete-time system



- Fold (generic)
- Transcritical (symmetry)
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Simplest Bifurcations

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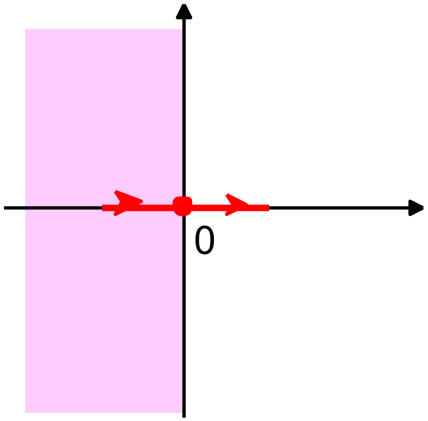
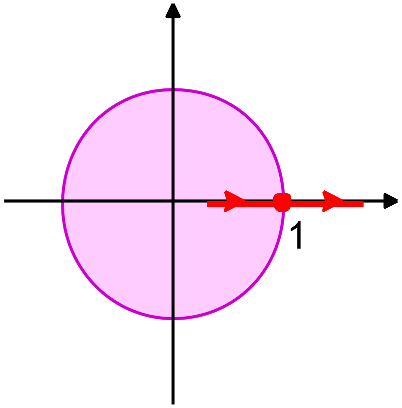
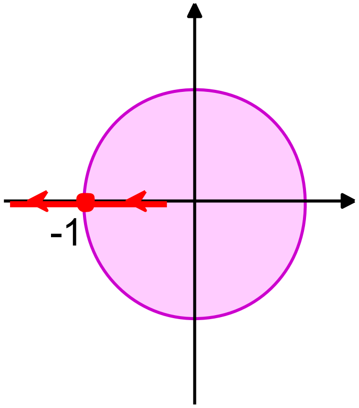
- Continuous-time system



- Fold (generic)
- Transcritical (symmetry)
- Pitchfork (symmetry)

- Discrete-time system





- Flip (period-doubling)

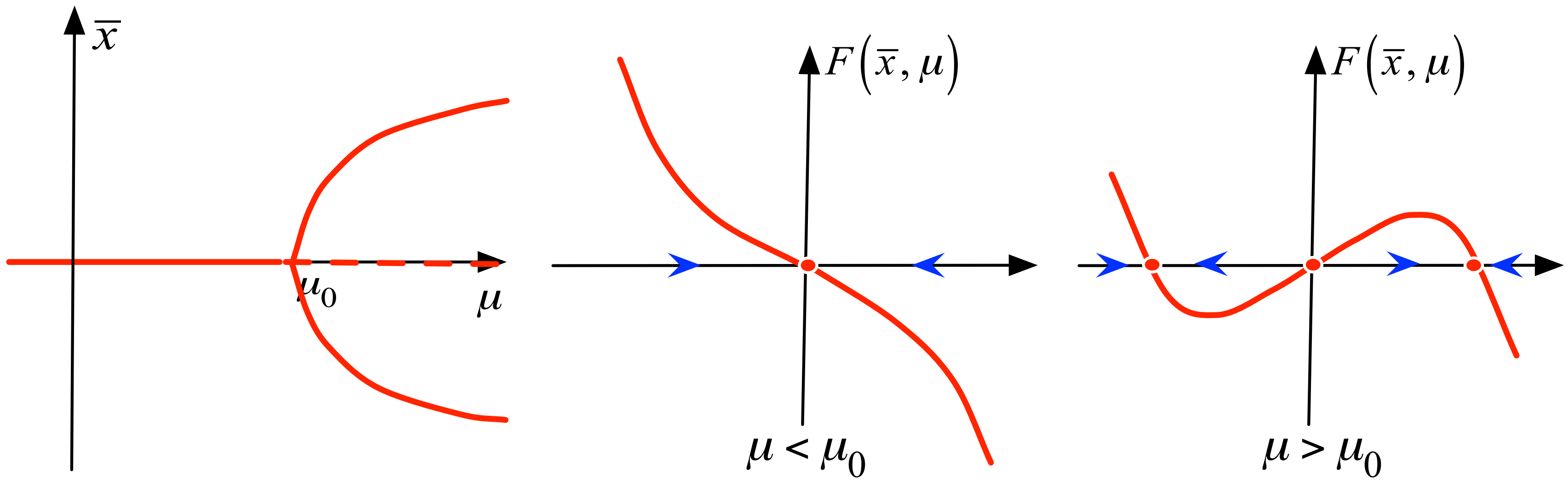
Flip Bifurcation

- $x(t+1) = F(x(t), \mu)$ with $F: \mathbb{R}^2 \to \mathbb{R}$ a C^3-function
- Let $(\bar{x}_0, \mu_0)$ be such that $F(\bar{x}_0, \mu_0) = \bar{x}_0$ and $\frac{\partial F}{\partial x}(\bar{x}_0, \mu_0) = -1$
- We consider directly the normal form $F(x, \mu) = -(1+\mu)x \pm x^3$
  - $(\bar{x}_0, \mu_0) = (0{,}0)$
- $F^{(2)}(x, \mu) = (F \circ F)(x, \mu) = -(1+\mu)[-(1+\mu)x \pm x^3] \pm [-(1+\mu)x \pm x^3]^3$

$$= \ldots = (1+\mu)^2 x \mp (1+\mu)(2+2\mu+\mu^2)x^3 - 3(1+\mu)^2 x^5 \mp 3(1+\mu)x^7 - x^9$$

- Now, $F^{(2)}(\bar{x}, \mu) = -F^{(2)}(-\bar{x}, \mu)$ is odd in $\bar{x}$ and
  - $\frac{\partial F^{(2)}}{\partial x}(0{,}0) = [(1+\mu)^2 \mp 3(1+\mu)(2+2\mu+\mu^2)x^2 + O(x^4)]_{(0,0)} = 1$
  - $\frac{\partial^2 F^{(2)}}{\partial x \partial \mu}(0{,}0) = [2(1+\mu) + O(x^2)]_{(0,0)} = 2$
  - $\frac{\partial^3 F^{(2)}}{\partial x^3}(0{,}0) = [\mp 6(1+\mu)(2+2\mu+\mu^2) + O(x^2)]_{(0,0)} = \mp 12$
- Therefore, the fixed point at the origin of $x(t+1) = F^{(2)}(x(t), \mu)$ undergoes a pitchfork bifurcation (subcritical if $F(x, \mu) = -(1+\mu)x - x^3$ and supercritical if $F(x, \mu) = -(1+\mu)x + x^3$)

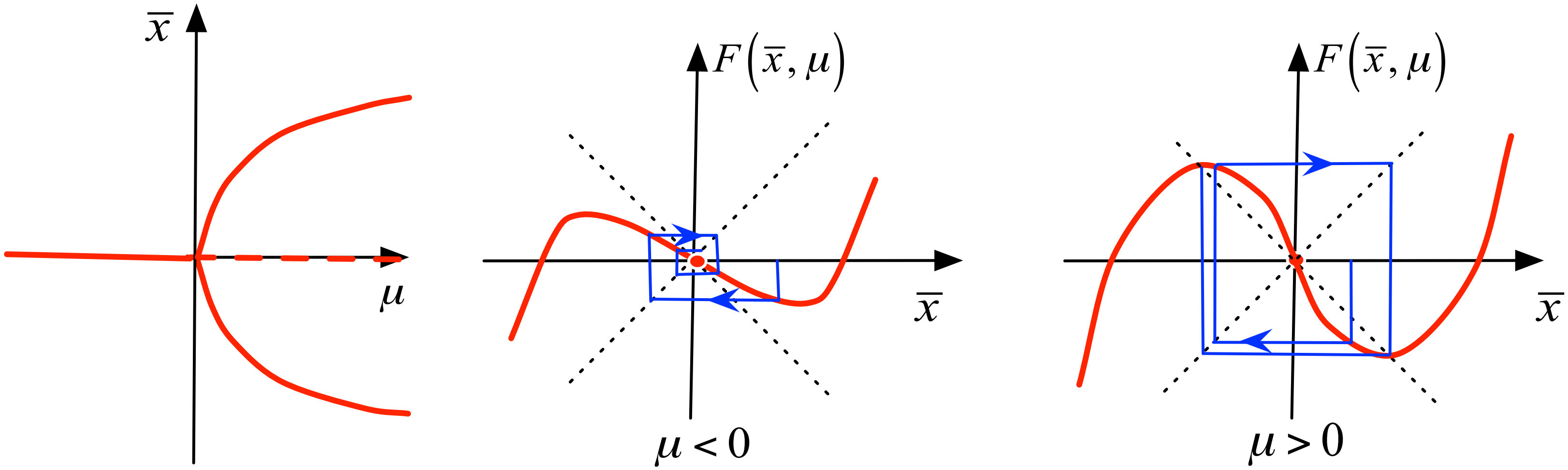
Remember: Pitchfork Bifurcation (supercritical)



Flip Bifurcation

- $x(t+1) = F(x(t),\mu) = -(1+\mu)x(t) \pm x^3(t)$;
- $F(0,0) = 0$ and $\frac{\partial F}{\partial x}(0,0) = -1$.
- The fixed point at the origin of $x(t+1) = F^{(2)}(x(t),\mu)$ undergoes a pitchfork bifurcation at $\mu = 0 \Rightarrow$ in a neighborhood of 0, this system has 3 fixed points for $\mu < 0$ (or $\mu > 0$), and 1 fixed point for $\mu > 0$ (resp., $\mu < 0$).
- Now, a fixed point of $x(t+1) = F^{(2)}(x(t),\mu)$ is either a fixed point or a 2-periodic solution of the original system $x(t+1) = F(x(t),\mu)$.
- Implicit function Theorem: Since $\frac{\partial F}{\partial x}(0,0) \neq 1$,
  - $\exists$ neighborhood $\mathcal{U} \subset \mathbb{R}^2$ of $(0,0)$
  - $\exists$ neighborhood $\mathcal{V} \subset \mathbb{R}$ of 0
  - $\exists$ (unique) C^1 function g: $\mathcal{V} \to \mathbb{R}$ with $g(0) = 0$ such that
  $$F(\bar{x},\mu) - \bar{x} = 0 \text{ for } (\bar{x},\mu) \in \mathcal{U} \Leftrightarrow \bar{x} = g(\mu) \text{ for } \mu \in \mathcal{V}$$
  - Moreover, $\frac{\partial g}{\partial \mu}(0) = -\left(\frac{\partial F}{\partial x}(0,0) - 1\right)^{-1} \cdot \frac{\partial F}{\partial \mu}(0,0) = -(-1-1)^{-1} \cdot 0 = 0$.
  - Hence the only fixed point of the original system in neighborhood $\mathcal{U}$ of $(0,0)$ is $\bar{x}(\mu) = 0$.
- Therefore the other two fixed points of $x(t+1) = F^{(2)}(x(t),\mu)$ are a 2-periodic solution of the original system $x(t+1) = F(x(t),\mu)$.

Flip Bifurcation



Theorem: Flip Bifurcation

- $x(t+1) = F(x(t), \mu)$ with $(\bar{x}_0, \mu_0)$ such that

$$\begin{aligned} F(\overline{x}_0, \mu_0) &= \overline{x}_0 \\ \frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) &= -1 \\ \left[\frac{\partial^2 F}{\partial \mu \partial x} + \frac{1}{2}\left(\frac{\partial F}{\partial \mu}\right)\left(\frac{\partial^2 F}{\partial x^2}\right)\right](\overline{x}_0, \mu_0) = \alpha &\neq 0 \\ \frac{1}{6}\frac{\partial^3 F}{\partial x^3}(\overline{x}_0, \mu_0) + \left(\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0)\right)^2 = \beta &\neq 0. \end{aligned}$$

- Then the system undergoes a flip bifurcation at $(\bar{x}_0, \mu_0)$, i.e. in a neighborhood of $(\bar{x}_0, \mu_0)$,

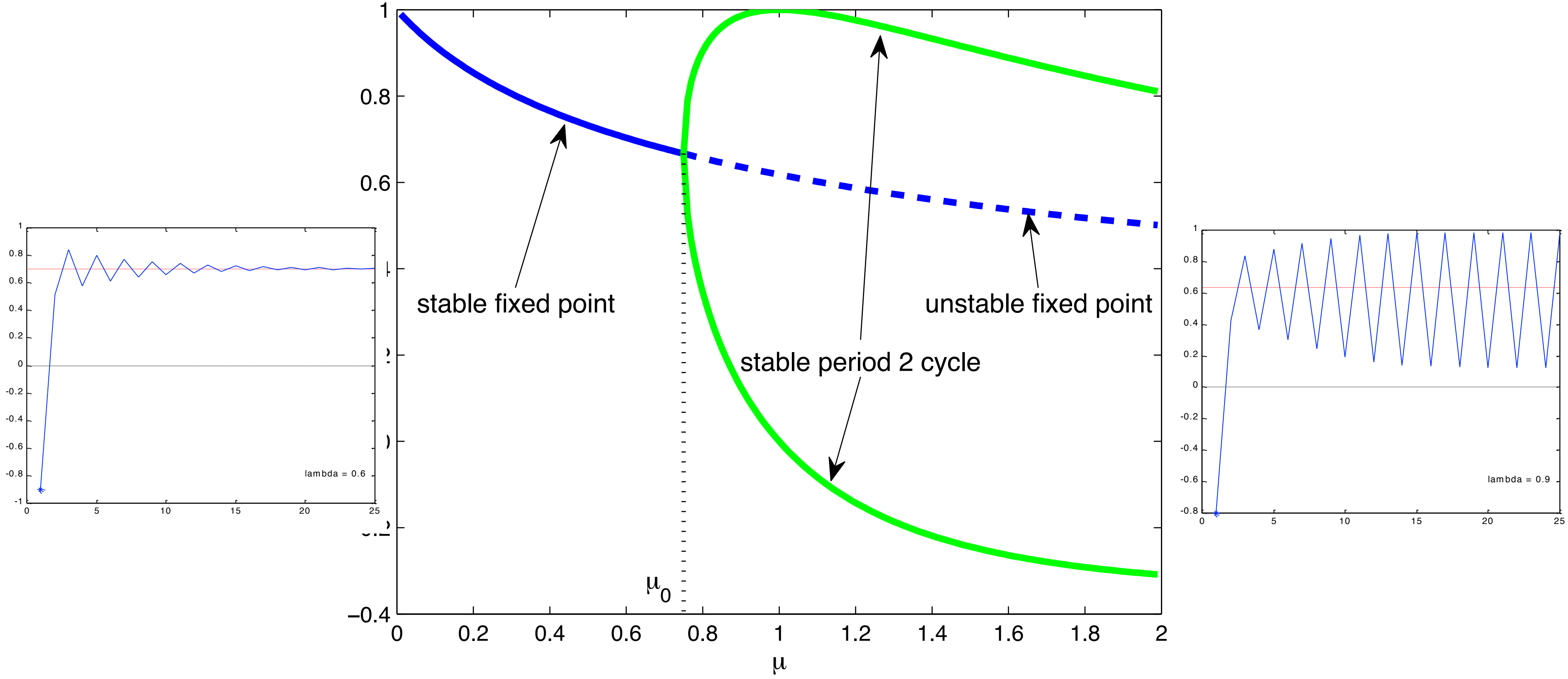
(i) for $\mu < \mu_0$, there is an asymptotically stable fixed point, whereas for $\mu > \mu_0$ the fixed point is unstable, and in addition, there is an asymptotically stable 2-cycle, or vice-versa (this is called a supercritical flip bifurcation*) or for $\mu < \mu_0$, there is an asymptotically stable fixed point and an unstable 2-cycle, whereas for $\mu > \mu_0$ there is only the fixed point and it is unstable, or vice-versa (this is called a* subcritical flip bifurcation*);*

(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

$$x(t+1) = -(1+\mu)x(t) \pm x^3(t). \tag{7.44}$$

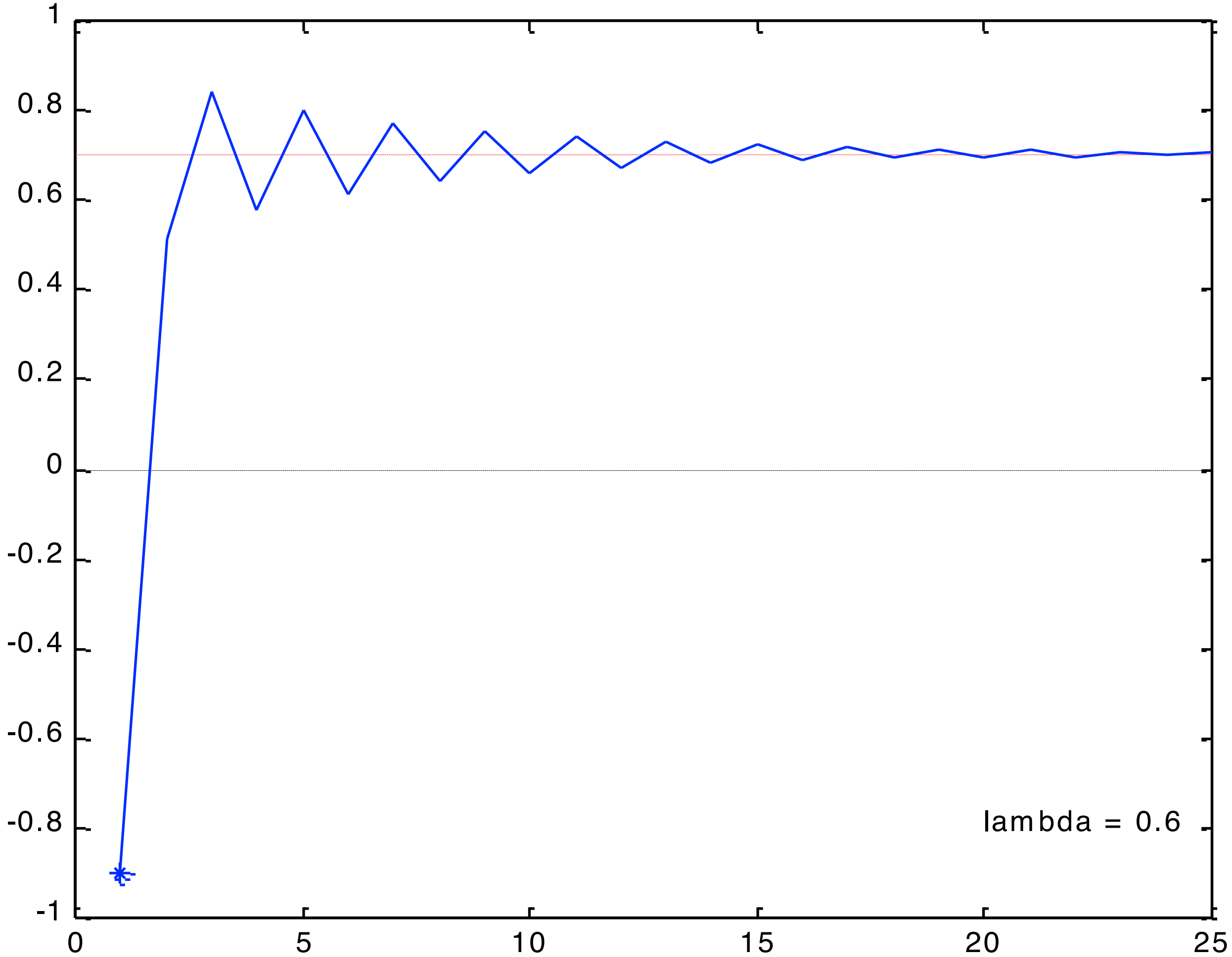
Example: Logistic Map

- $x(t+1) = 1 - \mu x^2(t)$



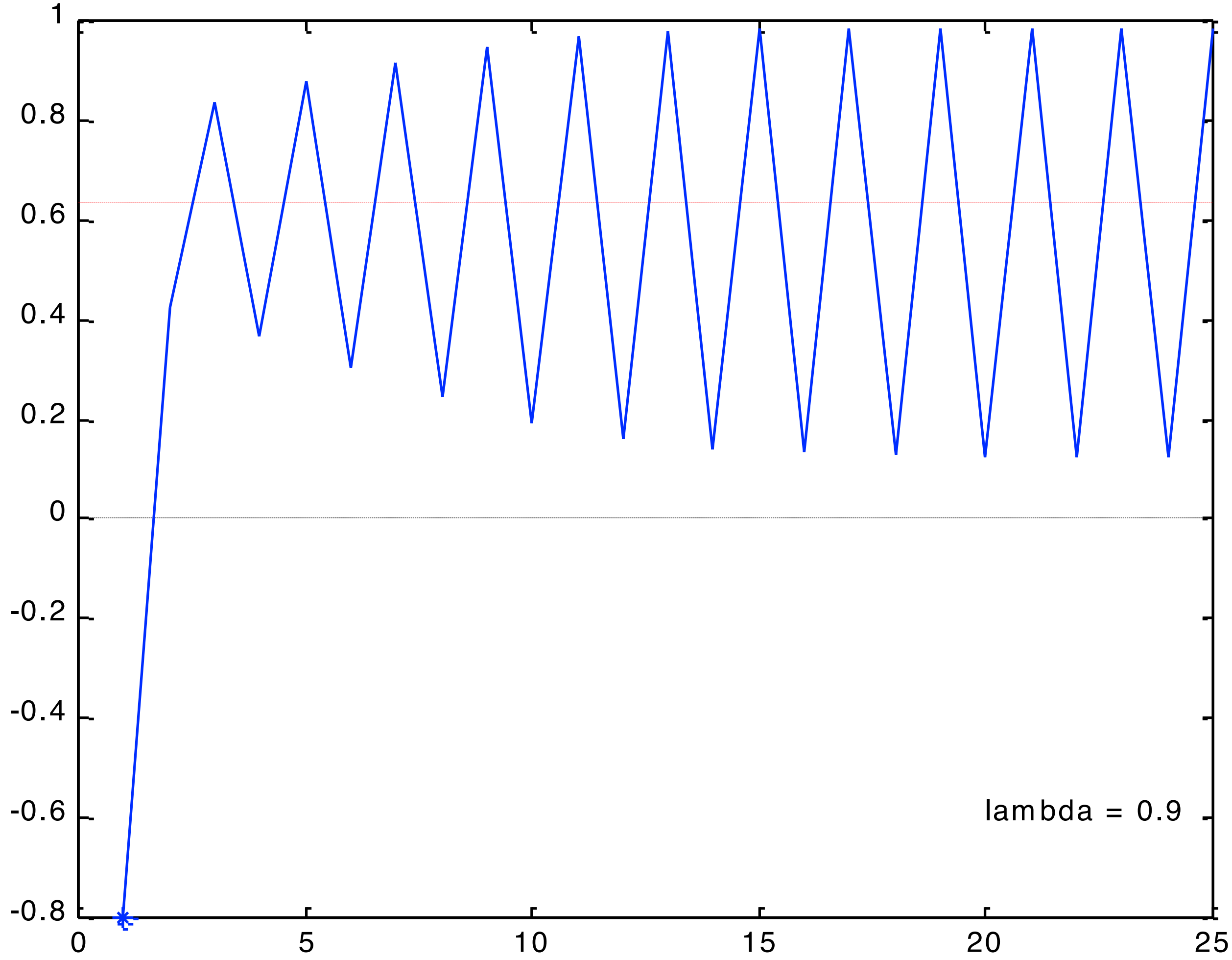
Example: Logistic Map

- $x(t+1) = 1-\mu x^2(t)$
- $\mu = 0.6$

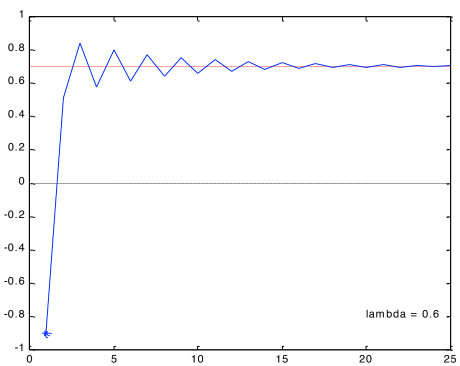


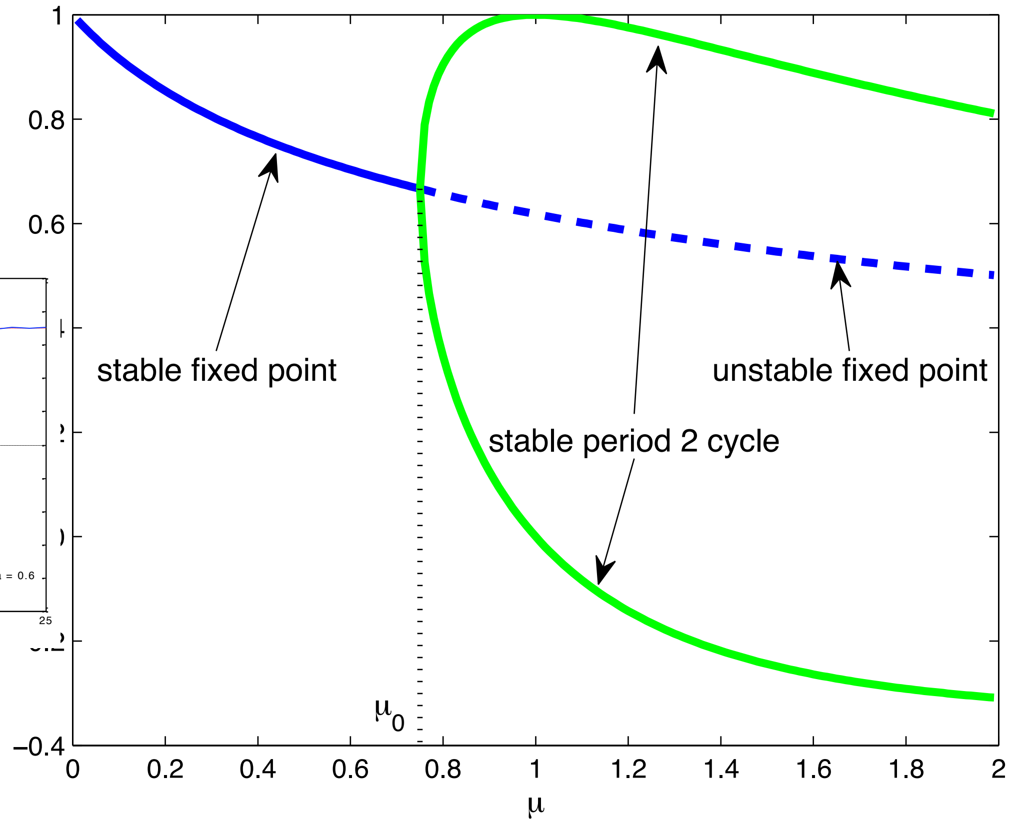
Example: Logistic Map

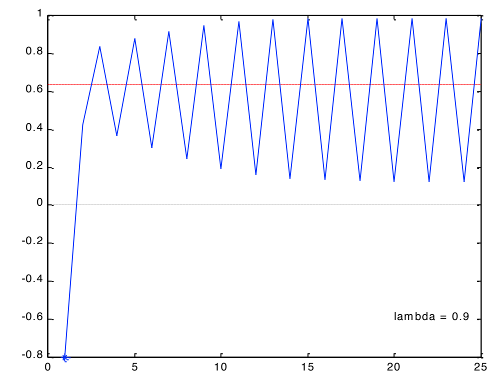
- $x(t+1) = 1-\mu x^2(t)$
- $\mu = 0.9$



Example: Logistic Map



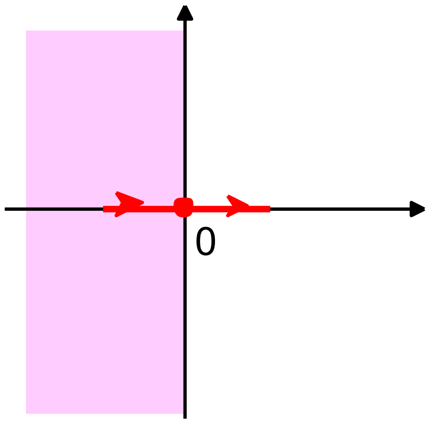


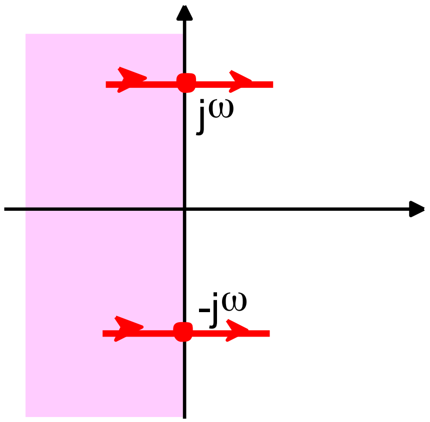


Simplest Bifurcations

- ❑ Bifurcations can only occur if Jacobian matrix ($\partial F/\partial x$) is non-hyperbolic

- ❑ Continuous-time system





- Fold (generic)
- Transcritical (symmetry)
- Pitchfork (symmetry)

- Andronov-Hopf (generic)

Andronov-Hopf Bifurcation

- $\dot{x}_1 = \mu x_1 - x_2 \pm x_1(x_1^2 + x_2^2)$
  $\dot{x}_2 = x_1 + \mu x_2 \pm x_2(x_1^2 + x_2^2)$
- $\bar{x} = (0,0)$ is an equilibrium point for all $\mu \in \mathbb{R}$
- $\frac{\partial F}{\partial x}((0,0), \mu\ ) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$
- $\bar{x} = (0,0)$ is a hyperbolic equilibrium point iff $\mu \neq 0$.
- Let $(\bar{x}_0, \mu_0) = ((0,0), 0)$. By the implicit function theorem, $\bar{x} = (0,0)$ is the only equilibrium point in neighborhood $\mathcal{U}$ of $(\bar{x}_0, \mu_0) = ((0,0), 0)$.
- From cartesian to polar coordinates

$$r = \sqrt{x_1^2 + x_2^2}$$

$$\varphi = \arctan\left(\frac{x_2}{x_1}\right)$$

- Equation in polar coordinates (r, φ)

$$\dot{r} = \mu r \pm r^3$$

$$\dot{\varphi} = 1$$

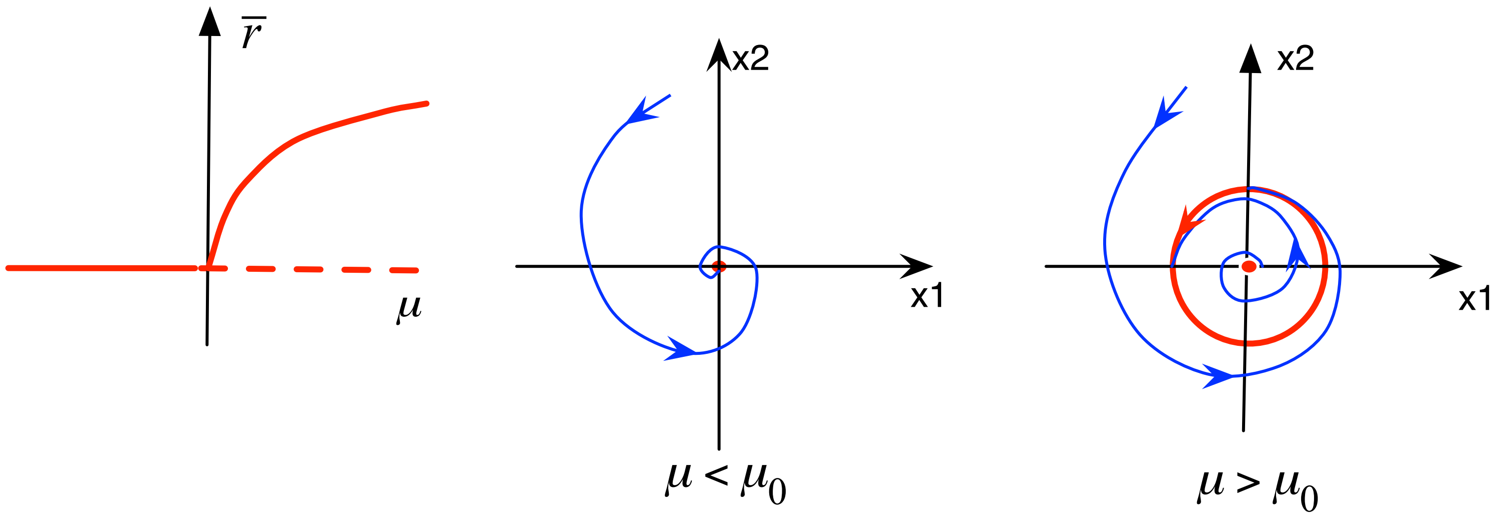
Andronov-Hopf Bifurcation

- $\dot{x}_1 = \mu x_1 - x_2 \pm x_1(x_1^2 + x_2^2)$
  $\dot{x}_2 = x_1 + \mu x_2 \pm x_2(x_1^2 + x_2^2)$
- Equation in polar coordinates (r, φ)

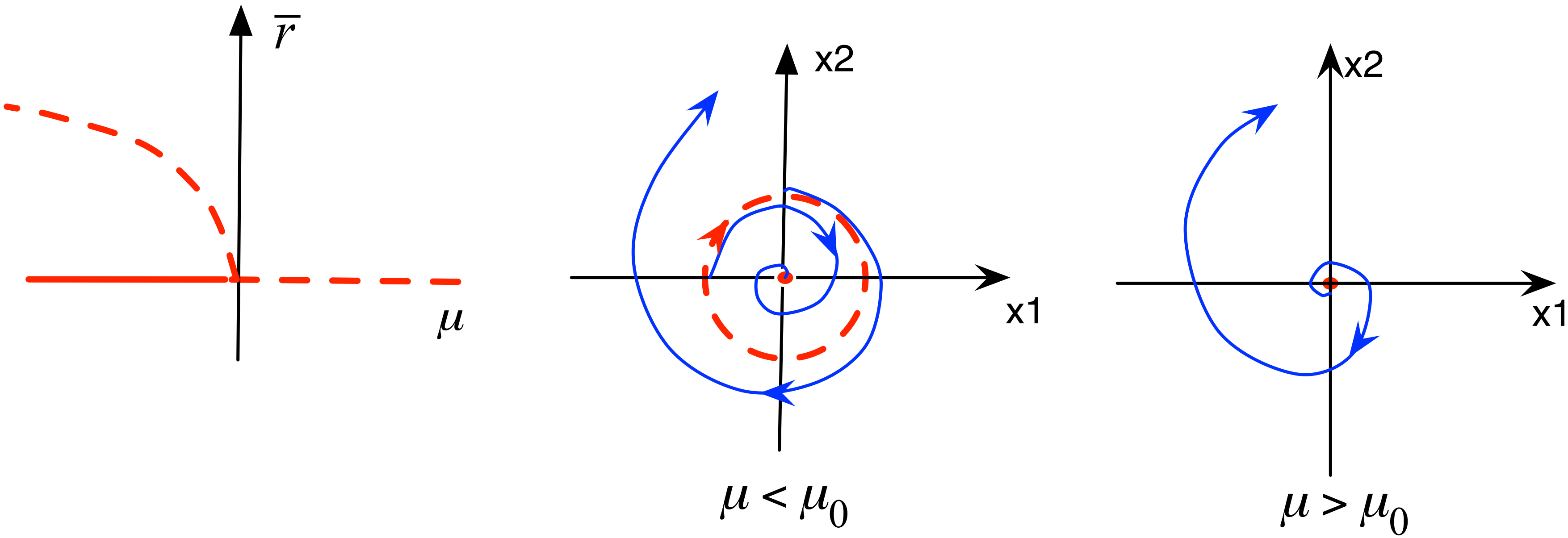
  $$\dot{r} = \mu r \pm r^3$$
  $$\dot{\varphi} = 1$$

- Normal form of a pitchfork bifurcation for r:
  - Subcritical for $\dot{r} = \mu r + r^3$
  - Supercritical for $\dot{r} = \mu r - r^3$
- Equilibrium points
  - $\bar{r} = 0$
  - $\bar{r}' = \sqrt{\pm\mu}$ (depending on the sign of μ)

Andronov-Hopf Bifurcation (supercritical)



Andronov-Hopf Bifurcation (subcritical)



Theorem: Andronov-Hopf Bifurcation

- $\dot{x} = F(x,\mu)$ with $(\bar{x}_0, \mu_0)$ such that $F(\bar{x}_0, \mu_0) = 0$ and $\frac{\partial F}{\partial x}(\bar{x}_0, \mu_0)$ has imaginary e-values $\pm j\omega_0$. Let $\lambda(\mu)$, $\lambda^*(\mu)$ be the e-values of $\frac{\partial F}{\partial x}(\bar{x}(\mu), \mu)$ in the neighborhood of $(\bar{x}_0, \mu_0)$. If
  - a complex non-degeneracy condition is satisfied,
  - and $\frac{d\Re(\lambda(\mu))}{d\mu}(\mu_0) \neq 0$
- Then the system undergoes an Andronov-Hopf bifurcation at $(\bar{x}_0, \mu_0)$, i.e. in a neighborhood of $(\bar{x}_0, \mu_0)$

(i) for $\mu < \mu_0$, there is an asymptotically stable equilibrium point $\overline{x}(\mu)$, whereas for $\mu > \mu_0$ the equilibrium point $\overline{x}(\mu)$ becomes unstable, and in addition, there is a stable periodic solution, or vice-versa (this is called a supercritical Andronov-Hopf bifurcation*) or for $\mu < \mu_0$, there is an asymptotically stable equilibrium point $\overline{x}(\mu)$ and an unstable periodic solution, whereas for $\mu > \mu_0$ there is only the equilibrium point $\overline{x}(\mu)$ and it is unstable, or vice-versa (this is called a* subcritical Andronov-Hopf bifurcation*);*

(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

$$\begin{aligned}\dot{x}_1 &= \mu x_1 - x_2 \pm x_1\left(x_1^2 + x_2^2\right)\\ \dot{x}_2 &= x_1 + \mu x_2 \pm x_2\left(x_1^2 + x_2^2\right)\end{aligned}$$

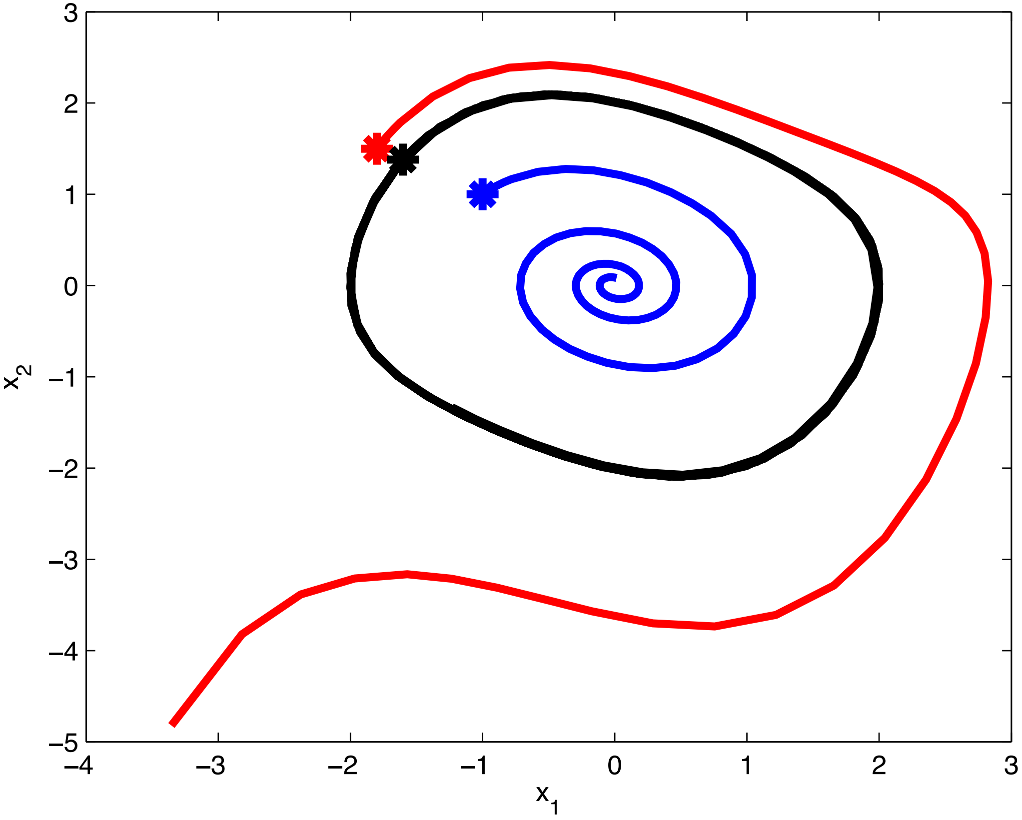
(iii) the period of the periodic solution is a differentiable function $T(\mu)$ of μ, with $T(\mu_0) = 2\pi/\omega_0$.

Van der Pol Oscillator

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - \lambda(x_1^2 - 1)x_2$$

- $\lambda = -0.3$



Van der Pol Oscillator

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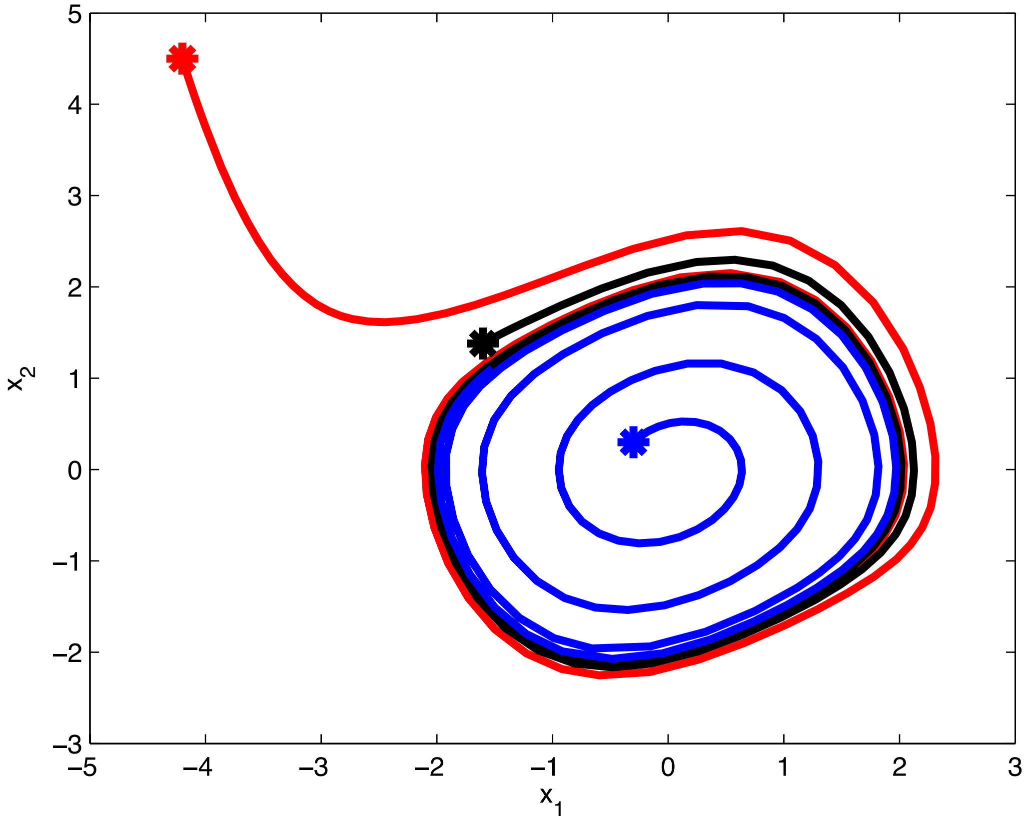
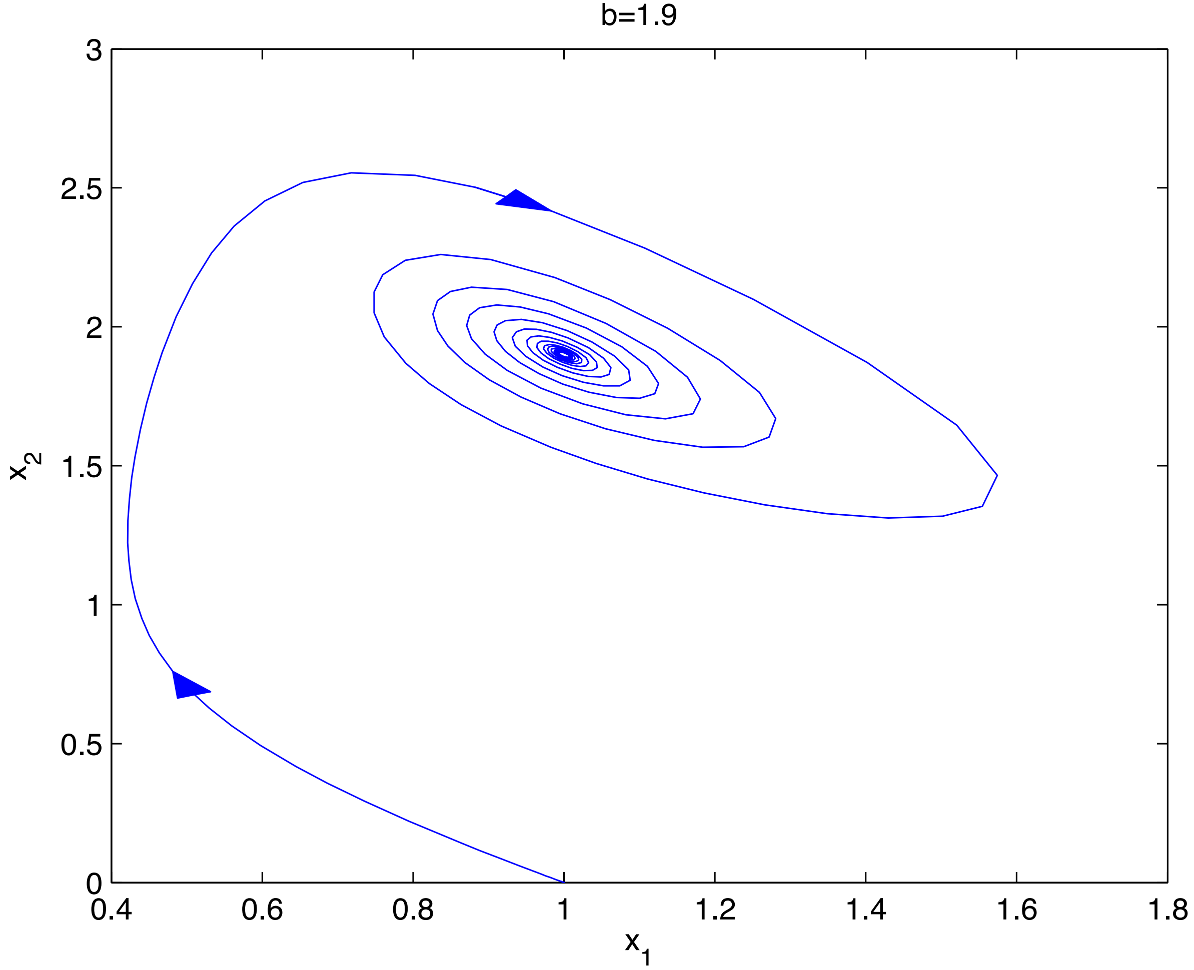


Brusselator

- $\dot{x}_1 = 1 - (b+1)x_1 + x_1^2 x_2$
- $\dot{x}_2 = bx_1 - x_1^2 x_2$

- $b = 1.9$



Brusselator

- $\dot{x}_1 = 1 - (b+1)x_1 + x_1^2 x_2$
- $\dot{x}_2 = bx_1 - x_1^2 x_2$

- $b = 2.1$