INDY

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Bifurcations

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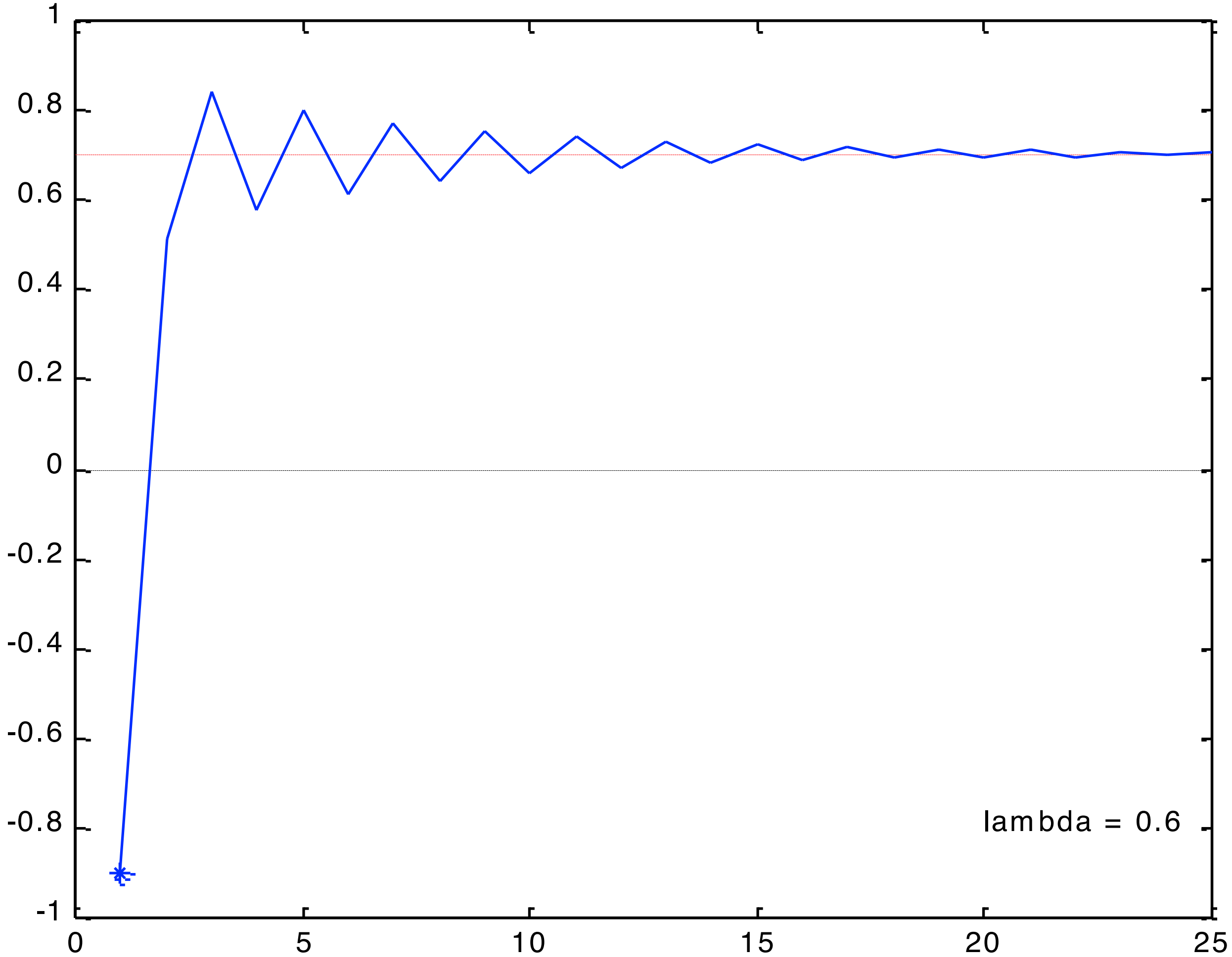
Definition

- Qualitative behavior of dynamical systems depend on parameters. Here we consider 1 parameter $\mu \in \mathbb{R}$
- Make parameter dependence explicit :
  - $\dot{x} = F(x) \longrightarrow \dot{x} = F(x, \mu)$
  - $x(t+1) = F(x(t)) \longrightarrow x(t+1) = F(x(t), \mu)$
  - $F: \mathbb{R}^{n+1} \to \mathbb{R}^n$ continuously differentiable (at least C^1)
- The system undergoes a bifurcation at μ_0 if there is no neighborhood $\mathcal{V} \subset \mathbb{R}$ of μ_0 such that all systems with $\mu \in \mathcal{V}$ have the same qualitative behavior.
- Same qualitative behavior ≡ there is a continuous coordinate and time transformation mapping the solutions of one system to the solutions of the other, and vice versa.
- Codimension of a bifurcation = number of parameters that must be varied for the bifurcation to occur.
  - Here we consider only bifurcations of codimension 1 ($\mu \in \mathbb{R}$)

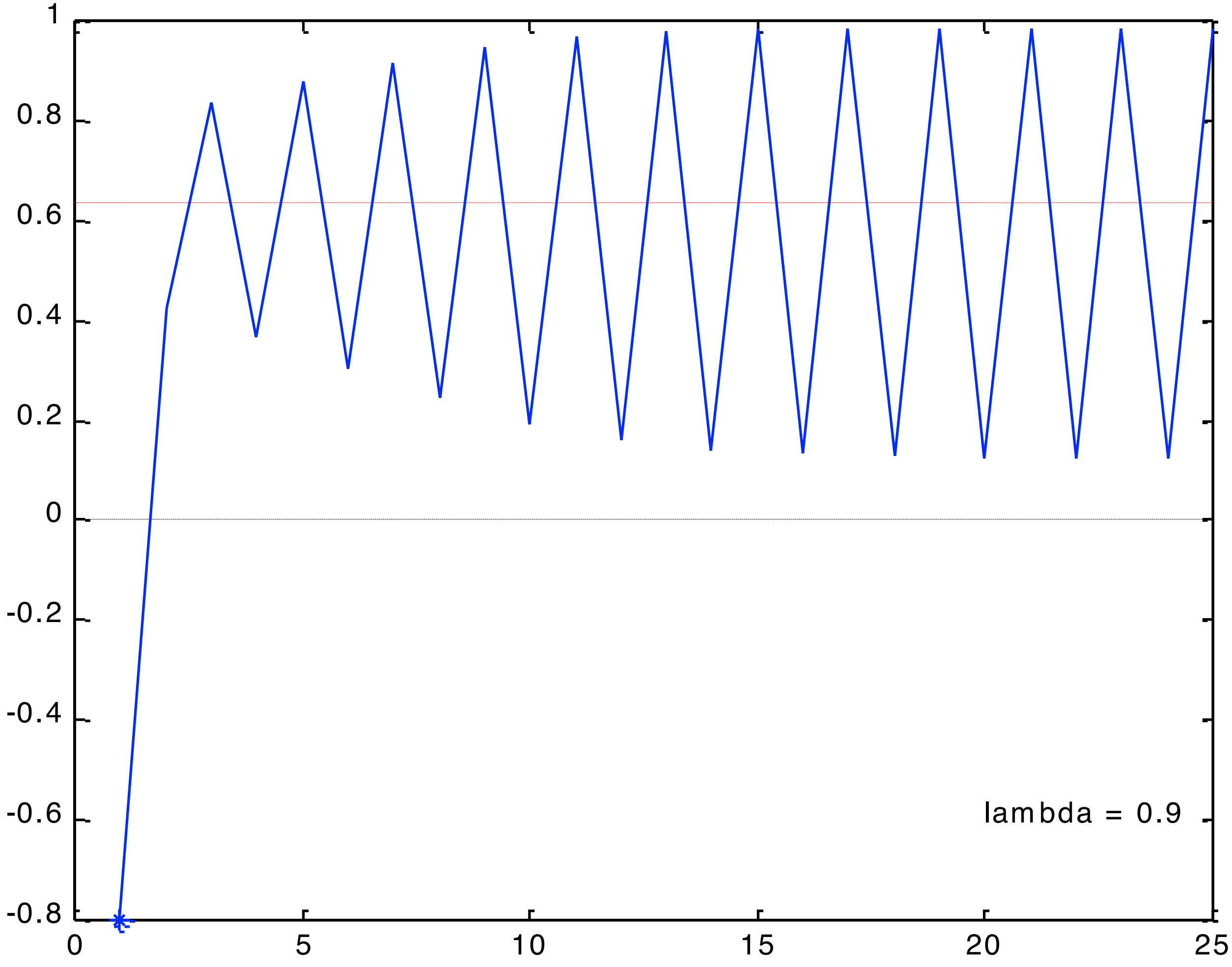
Example: Logistic Map

- $x(t+1) = 1-\mu x^2(t)$
- $\mu = 0.6$

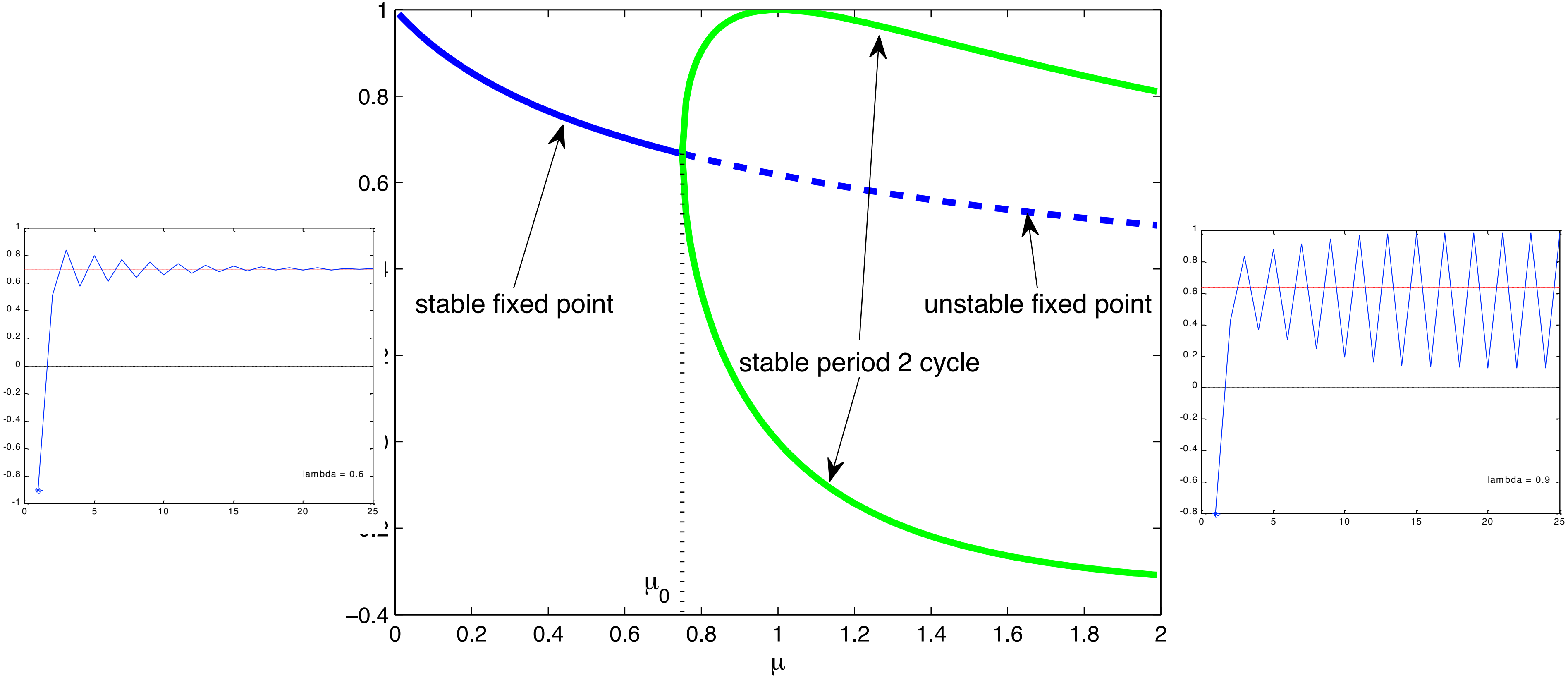


Example: Logistic Map

- $x(t+1) = 1-\mu x^2(t)$
- $\mu = 0.9$



Example: Logistic Map



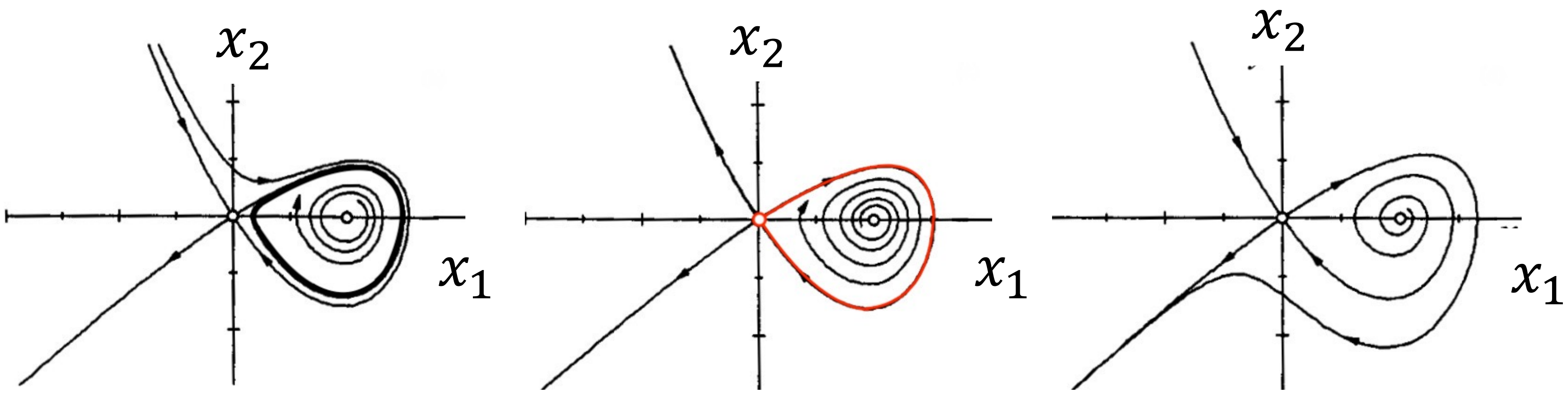
Local vs global bifurcation

- Local bifurcation: the qualitative change of asymptotic behavior takes place locally around a solution (in the course, an equilibrium point or a fixed point).
  - Examples:
    - Fold (saddle-node) bifurcation
    - Transcritical bifurcation
    - Pitchfork bifurcation
    - Flip (period-doubling) bifurcation
    - Hopf bifurcation
- Global bifurcation: the qualitative change of asymptotic behavior does not take place locally around a solution
  - Examples:
    - Homoclinic bifurcation
    - Heteroclinic bifurcation
  - Not in this course.

Example of Global Bifurcation

- Example of Homoclinic bifurcation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \mu x_2 + x_1 - x_1^2 + x_1 x_2$$



$\mu < \mu_0$ $\quad$ $\mu = \mu_0 \approx -0.8645$ $\quad$ $\mu > \mu_0$

(From S. Strogatz,2015, p. 266; D. Kartofelev, Lecture notes 7, 2020)

Local Bifurcations of Equilibrium/Fixed Points

- Codimension of the bifurcation = 1 ($\mu \in \mathbb{R}$)
- Qualitative change of asymptotic behavior takes place locally in a neighborhood of an equilibrium (fixed) point.
- Existence for $\mu < \mu_0$ and non-existence for $\mu > \mu_0$ or vice-versa.
  - Does not occur if the implicit function theorem holds.
- Change of (local) stability between $\mu < \mu_0$ and $\mu > \mu_0$

Implicit Function Theorem

Theorem 7.1 (Implicit Function Theorem)**.** *Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be a C^1-function and suppose that*

$$F(x_0, y_0) = 0 \tag{7.6}$$

with $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$. Suppose that the $n \times n$ Jacobian matrix of F with respect to x is

$$J_x(x_0, y_0) = \frac{\partial F}{\partial x}(x_0, y_0) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(x_0, y_0) & \frac{\partial F_1}{\partial x_2}(x_0, y_0) & \cdots & \frac{\partial F_1}{\partial x_n}(x_0, y_0) \\ \frac{\partial F_2}{\partial x_1}(x_0, y_0) & \frac{\partial F_2}{\partial x_2}(x_0, y_0) & \cdots & \frac{\partial F_2}{\partial x_n}(x_0, y_0)) \\ \vdots & & \ddots & \\ \frac{\partial F_n}{\partial x_1}(x_0, y_0) & \frac{\partial F_n}{\partial x_2}(x_0, y_0) & \cdots & \frac{\partial F_n}{\partial x_n}(x_0, y_0) \end{bmatrix} \tag{7.7}$$

is non-singular (i.e is invertible). Then there is a neighborhood $\mathcal{U}$ of (x_0, y_0) in $\mathbb{R}^{n+m}$, a neighborhood $\mathcal{V}$ of y_0 in $\mathbb{R}^m$ and a C^1- function $g : \mathcal{V} \to \mathbb{R}^n$ such that all solutions of $F(x, y) = 0$ in $\mathcal{U}$ are given by $x = g(y)$. Moreover,

$$\begin{aligned} \frac{\partial g}{\partial y}(y_0) &= -\left(\frac{\partial F}{\partial x}\right)^{-1}(x_0, y_0) \cdot \frac{\partial F}{\partial y}(x_0, y_0) \\ &= -J_x^{-1}(x_0, y_0) J_y(x_0, y_0). \end{aligned} \tag{7.8}$$

Example: Implicit Function Theorem

- Let $F: \mathbb{R}^2 \to \mathbb{R} \quad (x, y) \to F(x, y) = x^2 + y^2 - 1$
- Let (x_0, y_0) be such that $F(x_0, y_0) = 0$
- Implicit function Theorem:

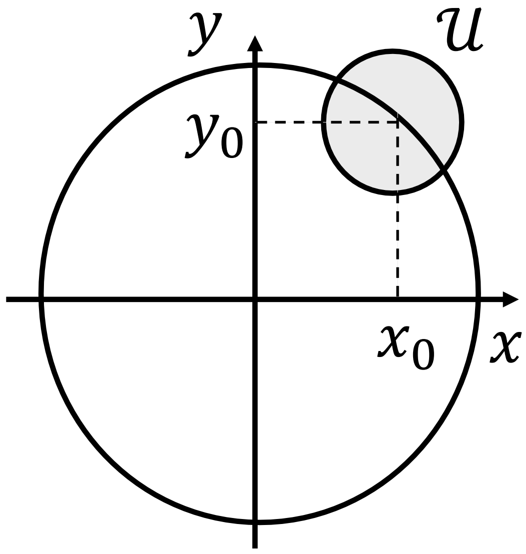
If $J_x(x_0, y_0) = \frac{\partial F}{\partial x}(x_0, y_0)$ is non singular then

- $\exists$ neighborhood $\mathcal{U} \subset \mathbb{R}^2$ of (x_0, y_0)
- $\exists$ neighborhood $\mathcal{V} \subset \mathbb{R}$ of y_0
- $\exists$ (unique) C[1] function $g: \mathcal{V} \to \mathbb{R}$ with $x_0 = g(y_0)$ and such that

$$F(x, y) = 0 \text{ for } (x, y) \in \mathcal{U} \Leftrightarrow x = g(y) \text{ for } y \in \mathcal{V}$$

- Moreover, $\frac{\partial g}{\partial y}(y_0) = -\left(\frac{\partial F}{\partial x}\right)^{-1}(x_0, y_0) \cdot \frac{\partial F}{\partial y}(x_0, y_0)$.

Example: Implicit Function Theorem



- Let $F: \mathbb{R}^2 \to \mathbb{R} \quad (x, y) \to F(x, y) = x^2 + y^2 - 1$
- Let (x_0, y_0) be s.t. $F(x_0, y_0) = x_0^2 + y_0^2 - 1 = 0$
- Implicit function Theorem:

If $J_x(x_0, y_0) = \frac{\partial F}{\partial x}(x_0, y_0) = 2x_0 \neq 0$ then

- $\exists$ neighborhood $\mathcal{U} \subset \mathbb{R}^2$ of (x_0, y_0)
- $\exists$ neighborhood $\mathcal{V} \subset \mathbb{R}$ of y_0
- $\exists$ (unique) C^1 function $g: \mathcal{V} \to \mathbb{R}$ with $x_0 = g(y_0)$ and such that

  $x^2 + y^2 - 1$ for $(x, y) \in \mathcal{U} \Leftrightarrow x = g(y) = \sqrt{1 - y^2}$

- Moreover, $\frac{\partial g}{\partial y}(y_0) = -\left(\frac{\partial F}{\partial x}\right)^{-1}(x_0, y_0) \cdot \frac{\partial F}{\partial y}(x_0, y_0) =$

$$-(2x_0)^{-1} \cdot 2y_0 = -\frac{y_0}{x_0} = -\frac{y_0}{g(y_0)} = -\frac{y_0}{\sqrt{1-y_0^2}}$$

Application to Continuous Time System

- $\dot{x} = F(x, \mu)$ with $F: \mathbb{R}^{n+1} \to \mathbb{R}^n$ continuously differentiable (C^1)
- Any equilibrium point $\bar{x}$ satisfies $F(\bar{x}, \mu) = 0$
- Let $(\bar{x}_0, \mu_0)$ be such that $F(\bar{x}_0, \mu_0) = 0$
- When can we write $\bar{x} = g(\mu)$ in a neighborhood $\mathcal{V}$ of μ_0 with g a C^1 function?
- Implicit function Theorem:

If $J_x(\bar{x}_0, \mu_0) = \frac{\partial F}{\partial x}(\bar{x}_0, \mu_0)$ is non singular (i.e., all its eigenvalues are non-zero), then

- $\exists$ neighborhood $\mathcal{U} \subset \mathbb{R}^2$ of $(\bar{x}_0, \mu_0)$
- $\exists$ neighborhood $\mathcal{V} \subset \mathbb{R}$ of μ_0
- $\exists$ (unique) C^1 function $g: \mathcal{V} \to \mathbb{R}$ with $\bar{x}_0 = g(\mu_0)$ and such that

$$F(\bar{x}, \mu) = 0 \text{ for } (\bar{x}, \mu) \in \mathcal{U} \Leftrightarrow \bar{x} = g(\mu) \text{ for } \mu \in \mathcal{V}$$

- Moreover, $\frac{\partial g}{\partial \mu}(\mu_0) = -\left(\frac{\partial F}{\partial x}\right)^{-1}(\bar{x}_0, \mu_0) \cdot \frac{\partial F}{\partial \mu}(\bar{x}_0, \mu_0)$.

Application to Discrete Time System

- $x(t+1) = F(x(t), \mu)$ with $F: \mathbb{R}^{n+1} \to \mathbb{R}^n$ cont. different. (C^1)
- Any fixed point $\bar{x}$ satisfies $F(\bar{x}, \mu) - \bar{x} = 0$
- Let $(\bar{x}_0, \mu_0)$ be such that $F(\bar{x}_0, \mu_0) - \bar{x}_0 = 0$
- When can we write $\bar{x} = g(\mu)$ in a neighborhood $\mathcal{V}$ of μ_0 with g a C^1 function?
- Implicit function Theorem:

If $J_x(\bar{x}_0, \mu_0) - I_n = \frac{\partial F}{\partial x}(\bar{x}_0, \mu_0) - I_n$ is non singular (i.e., all the eigenvalues of $J_x(\bar{x}_0, \mu_0)$ are not equal to 1), then

- ∃ neighborhood $\mathcal{U} \subset \mathbb{R}^2$ of $(\bar{x}_0, \mu_0)$
- ∃ neighborhood $\mathcal{V} \subset \mathbb{R}$ of μ_0
- ∃ (unique) C^1 function $g: \mathcal{V} \to \mathbb{R}$ with $\bar{x}_0 = g(\mu_0)$ and such that

$$F(\bar{x}, \mu) - \bar{x} = 0 \text{ for } (\bar{x}, \mu) \in \mathcal{U} \Leftrightarrow \bar{x} = g(\mu) \text{ for } \mu \in \mathcal{V}$$

- Moreover, $\frac{\partial g}{\partial \mu}(\mu_0) = -\left(\frac{\partial F}{\partial x} - I_n\right)^{-1}(\bar{x}_0, \mu_0) \cdot \frac{\partial F}{\partial \mu}(\bar{x}_0, \mu_0)$.

Necessary Condition for Bifurcation

- Implicit equation:
  - (CT) Equilibrium point equation: $F(\bar{x}, \mu) = 0$.
  - (DT) Fixed point equation $F(\bar{x}, \mu) - \bar{x} = 0$.
- If Jacobian matrix (∂F/∂x) does not have
  - (CT) the eigenvalue 0
  - (DT) the eigenvalue 1
  - then in a neighborhood of $(\bar{x}_0, \mu_0)$, the equilibrium/fixed points are given by a continuously differentiable 1-parameter family $\bar{x}(\mu)$ with

$$\overline{x}(\mu_0) = \overline{x}_0$$

and

$$\frac{\partial \overline{x}}{\partial \mu}(\mu_0) = -\left(\frac{\partial F}{\partial x}\right)^{-1}(\overline{x}_0, \mu_0) \cdot \frac{\partial F}{\partial \mu}(\overline{x}_0, \mu_0).$$

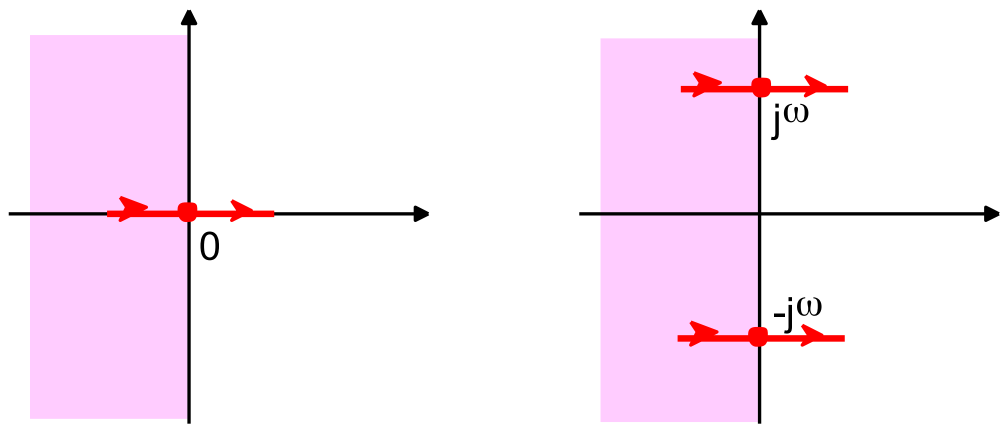
- A local bifurcation might however still occurs, if the local stability of the equilibrium/fixed point changes at $\mu = \mu_0$.
- If Jacobian matrix (∂F/∂x) is hyperbolic, i.e., does not have
  - (CT) eigenvalues on the imaginary axis
  - (DT) eigenvalues on the unit circle

  then there is no bifurcation at μ_0.

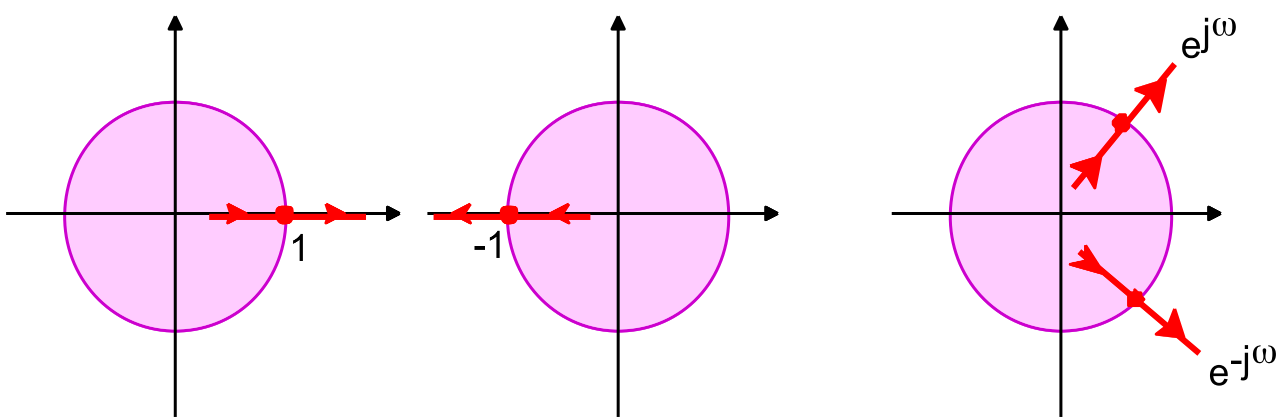
Simplest Bifurcations

- Bifurcations can only occur if Jacobian matrix ($\partial F/\partial x$) is non-hyperbolic

- Continuous-time system



- Discrete-time system



Theorem: Fold Bifurcation

- $\dot{x} = F(x, \mu)$ with $(\bar{x}_0, \mu_0)$ such that

$$
\begin{aligned}
F(\overline{x}_0, \mu_0) &= 0 \\
\frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) &= 0 \\
\frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0) &\neq 0 \\
\frac{\partial F}{\partial \mu}(\overline{x}_0, \mu_0) &\neq 0.
\end{aligned}
$$

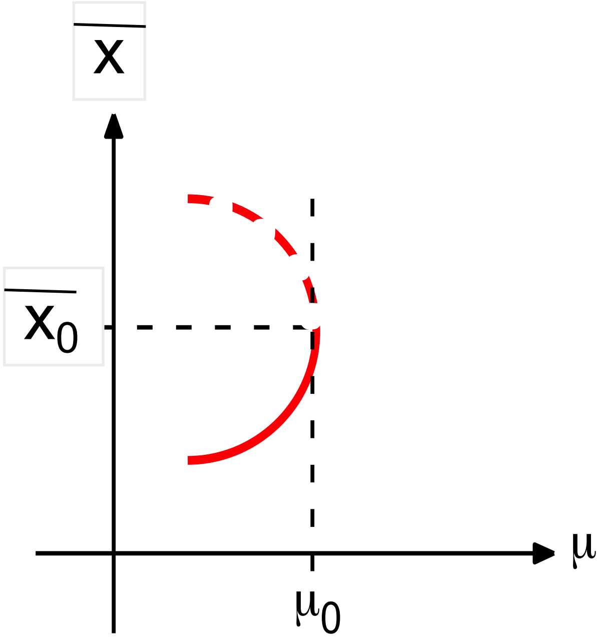
- Then the system undergoes a fold bifurcation $(\bar{x}_0, \mu_0)$, i.e. in a neighborhood of $(\bar{x}_0, \mu_0)$

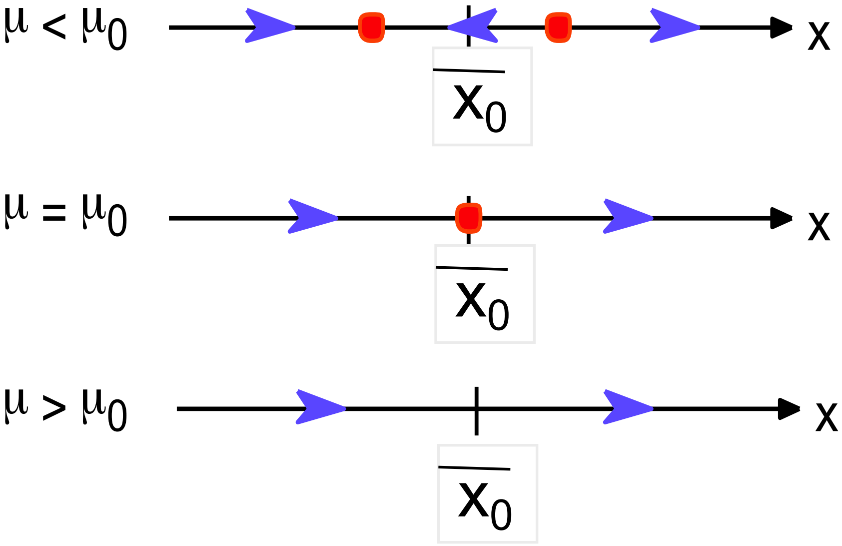
(i) for $\mu < \mu_0$, there are two equilibrium/fixed points, one asymptotically stable, the other unstable, and for $\mu > \mu_0$ there is none, or vice-versa;

(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

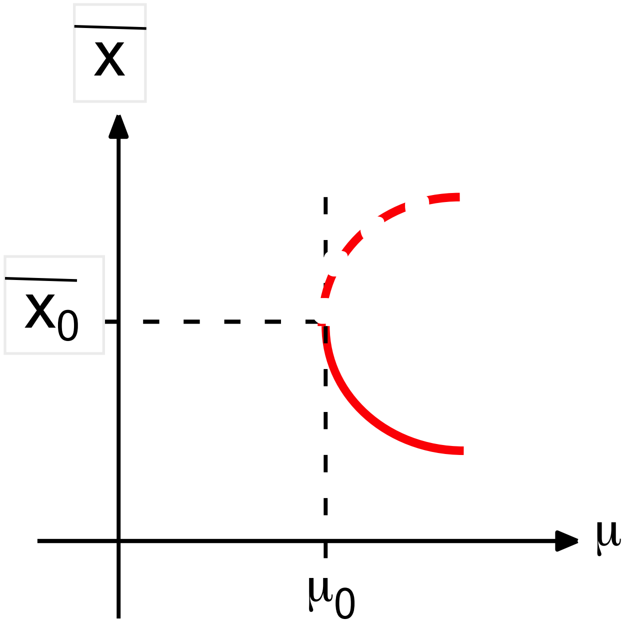
$$\dot{x}(t) = \mu \pm x^2(t) \tag{7.21}$$

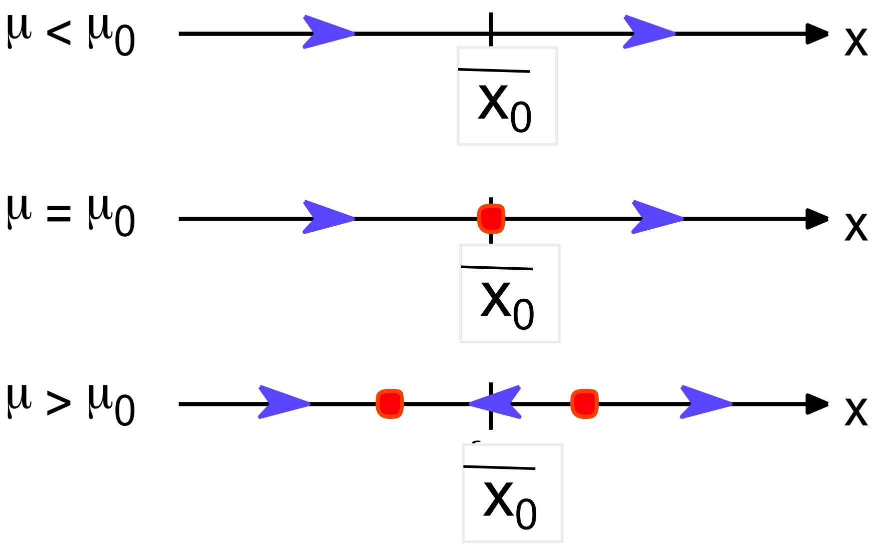
Fold Bifurcation: a > 0, b > 0



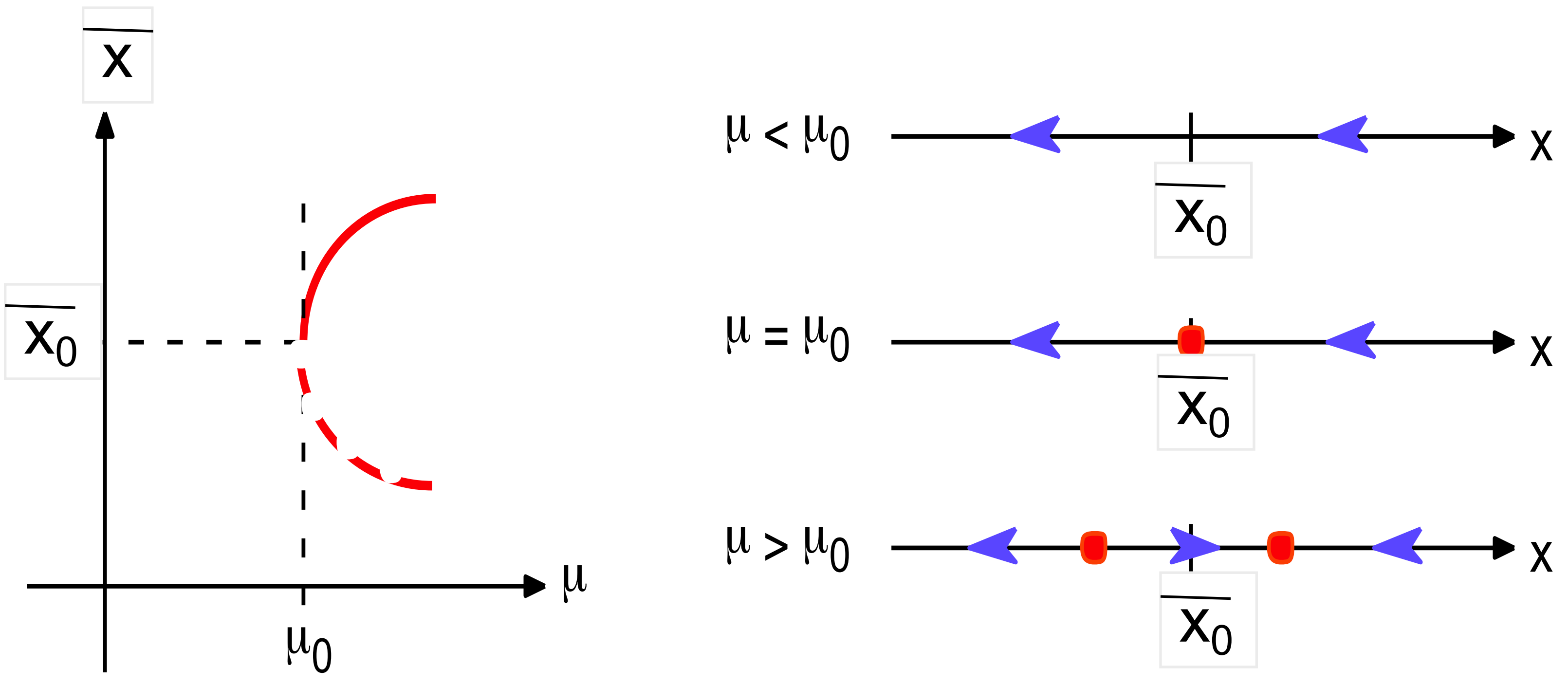


Fold Bifurcation: a > 0, b < 0

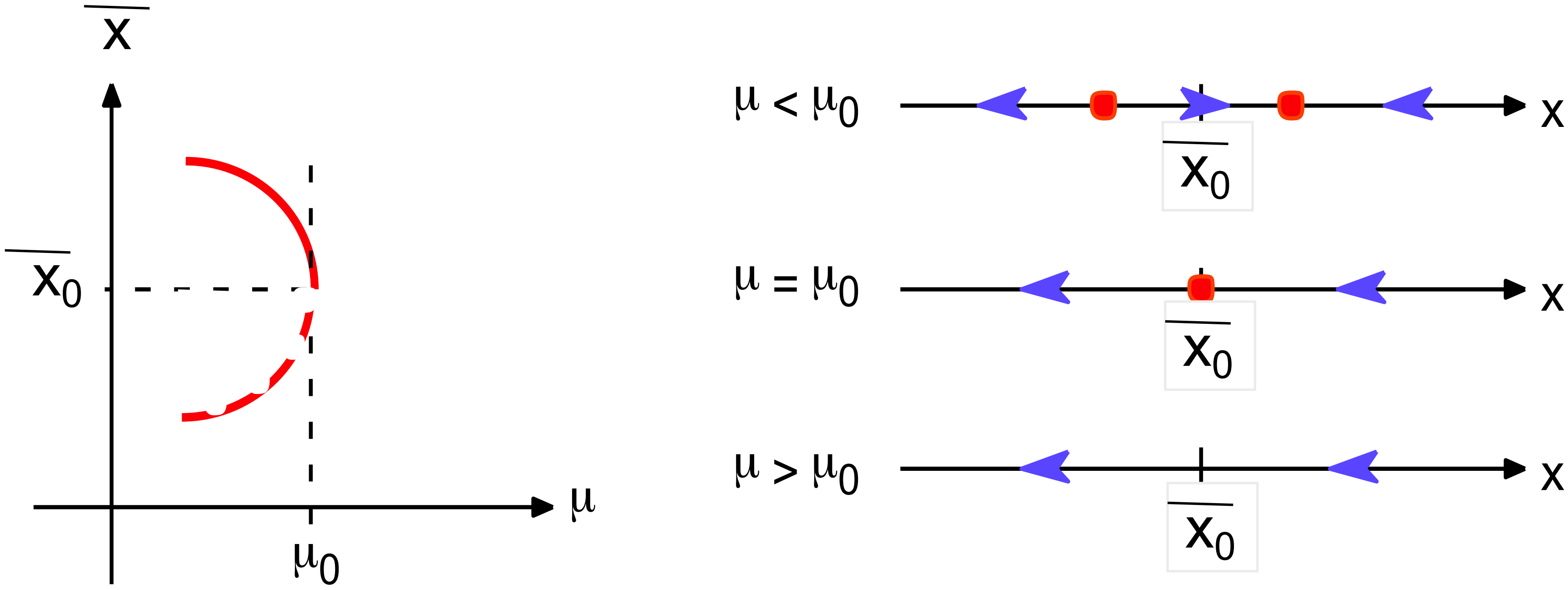




Fold Bifurcation: a < 0, b > 0



Fold Bifurcation: a < 0, b < 0



Theorem: Transcritical Bifurcation

- $\dot{x} = F(x, \mu)$ with $(\bar{x}_0, \mu_0)$ such that

$$
\begin{aligned}
F(\bar{x}_0, \mu_0) &= 0 \\
\frac{\partial F}{\partial x}(\bar{x}_0, \mu_0) &= 0 \\
\frac{\partial F}{\partial \mu}(\bar{x}_0, \mu_0) &= 0 \\
\frac{\partial^2 F}{\partial x^2}(\bar{x}_0, \mu_0) &\neq 0 \\
\left[\frac{\partial^2 F}{\partial \mu \partial x}\bar{x}_0, \mu_0)\right]^2 - \frac{\partial^2 F}{\partial x^2}(\bar{x}_0, \mu_0)\frac{\partial^2 F}{\partial \mu^2}(\bar{x}_0, \mu_0) &> 0
\end{aligned}
$$

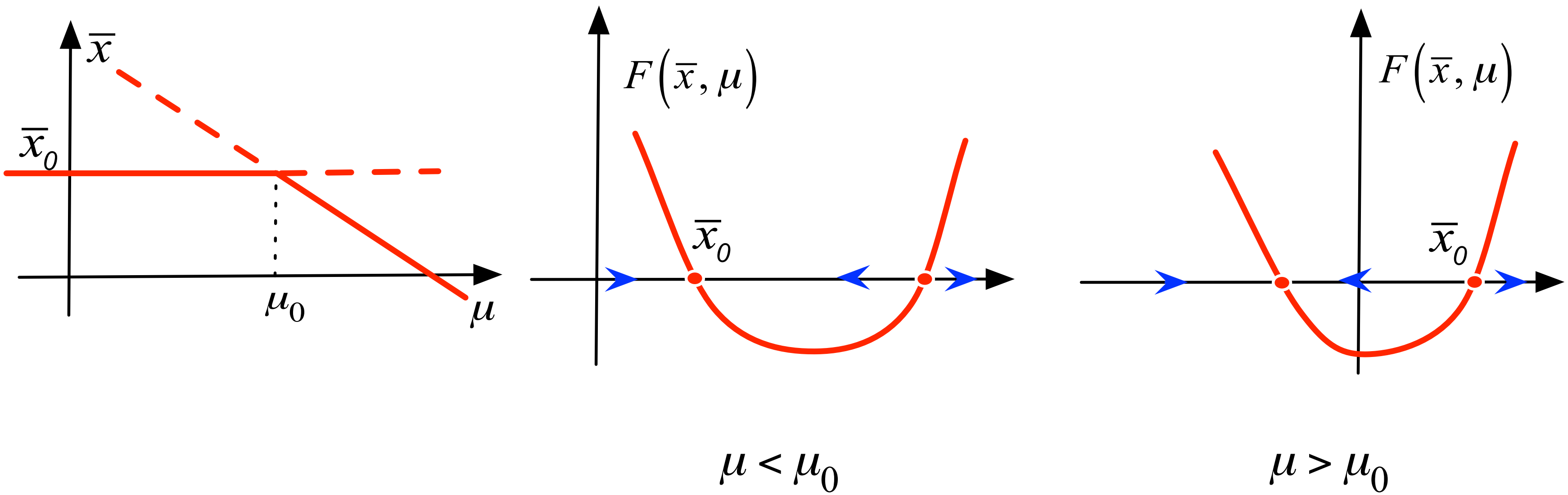
- Then the system undergoes a transcritical bifurcation at $(\bar{x}_0, \mu_0)$, i.e., in a neighborhood of $(\bar{x}_0, \mu_0)$

(i) for $\mu \neq \mu_0$, there are two equilibrium/fixed points, one asymptotically stable, the other unstable. They switch stability at $\mu = \mu_0$;

(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

$$\dot{x}(t) = \mu x(t) \pm x^2(t) \tag{7.37}$$

Transcritical Bifurcation: a > 0, b > 0



Theorem: Pitchfork Bifurcation

- $\dot{x} = F(x,\mu)$ with $F(x,\mu) = -F(-x,\mu)$ and $(\bar{x}_0,\mu_0) = (0,\mu_0)$ such that

$$\begin{aligned}\frac{\partial F}{\partial x}(0,\mu_0) &= 0\\ \frac{\partial^2 F}{\partial x \partial \mu}(0,\mu_0) &\neq 0\\ \frac{\partial^3 F}{\partial x^3}(0,\mu_0) &\neq 0.\end{aligned}$$

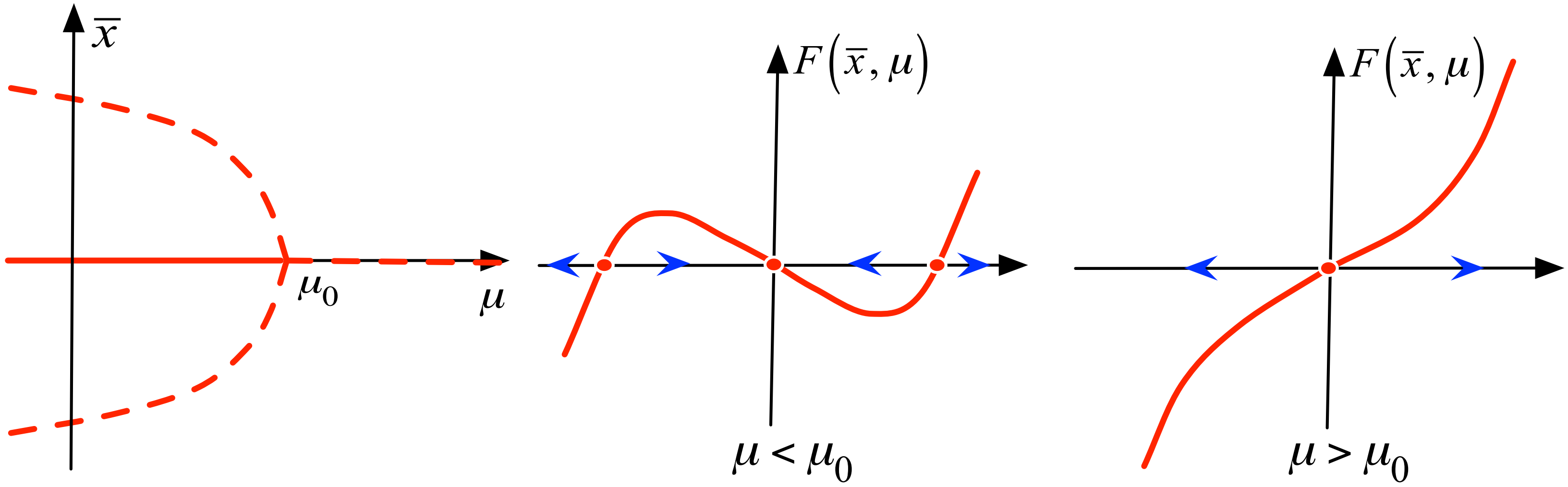
- Then the system undergoes a pitchfork bifurcation at $(0,\mu_0)$, i.e., in a neighborhood of $(0,\mu_0)$

(i) for $\mu < \mu_0$, the origin is the only equilibrium/fixed point and it is asymptotically stable, whereas for $\mu > \mu_0$ the origin is an unstable equilibrium/fixed point, and in addition, there are two asymptotically stable equilibrium/fixed points, or vice-versa (this is called a supercritical pitchfork bifurcation*) or for $\mu < \mu_0$, the origin is an asymptotically stable equilibrium/fixed point and in addition there are two unstable equilibrium/fixed points, whereas for $\mu > \mu_0$ the origin is the only equilibrium/fixed point and it is unstable, or vice-versa (this is called a* subcritical pitchfork bifurcation*);*

(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is

$$\dot{x}(t) = \mu x(t) \pm x^3(t) \tag{7.29}$$

Pitchfork Bifurcation: a > 0, b > 0 (subcritical)



Pitchfork Bifurcation: a > 0, b < 0 (supercritical)