INDY

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Chaos

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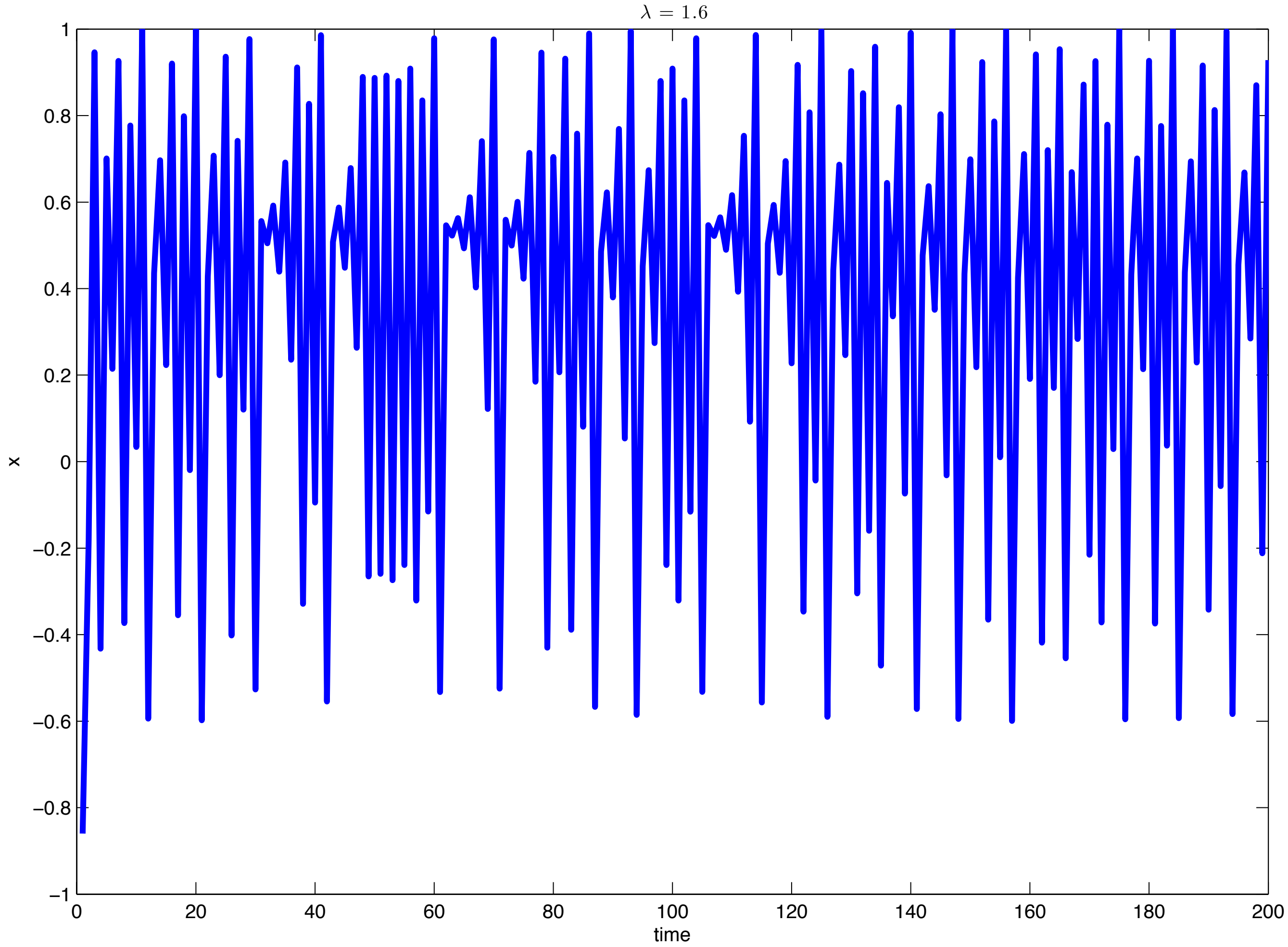
Property 1: Irregular trajectories

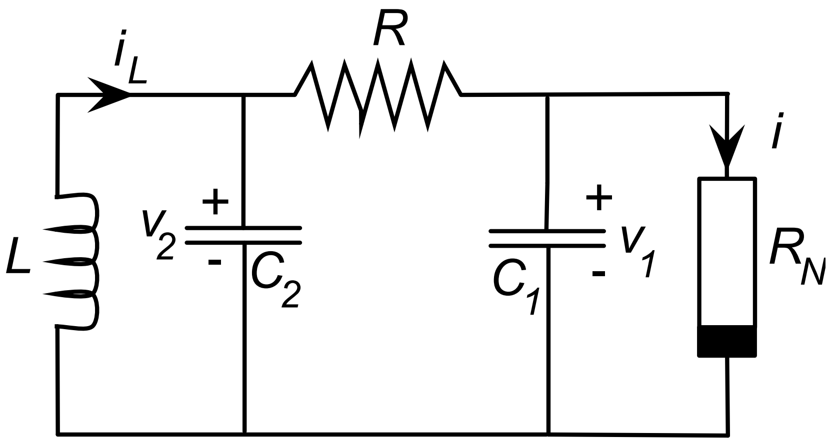
$$x(t+1) = 1 - \lambda x^2(t)$$

- ❑ $\lambda = 1.6$

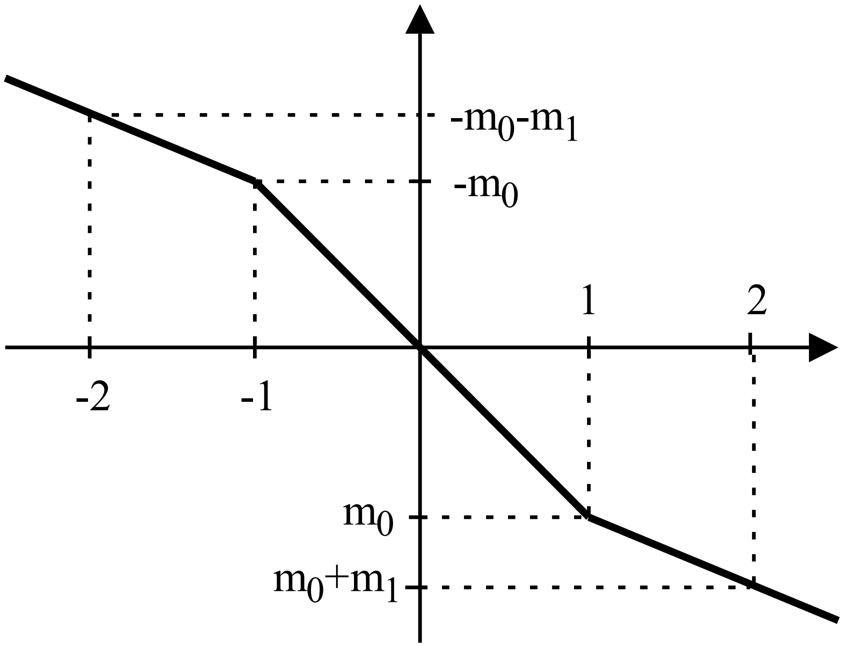


Chua's Circuit

$$\dot{x}_1 = \alpha(-x_1 - f(x_1) + x_2)$$
$$\dot{x}_2 = x_1 - x_2 + x_3$$
$$\dot{x}_3 = -\beta x_2$$



- $\alpha = 9$
- $\beta = 100/7$
- $m_0=-8/7$
- $m_1=-5/7$



Chua's Circuit

$$\dot{x}_1 = \alpha(-x_1 - f(x_1) + x_2)$$

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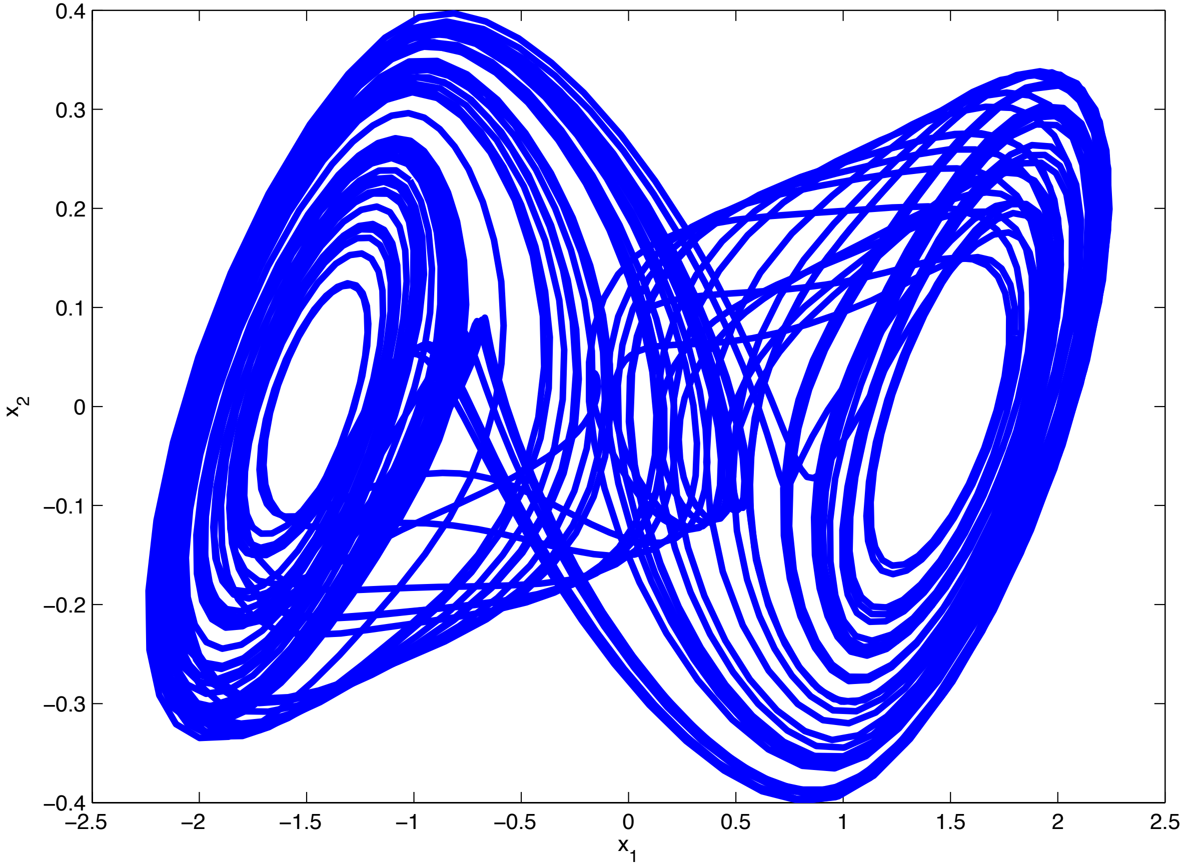

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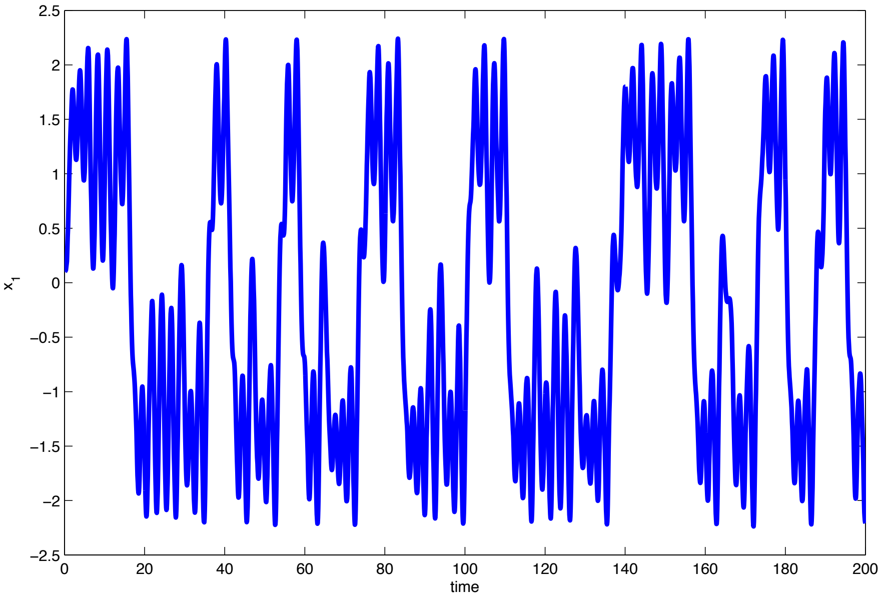

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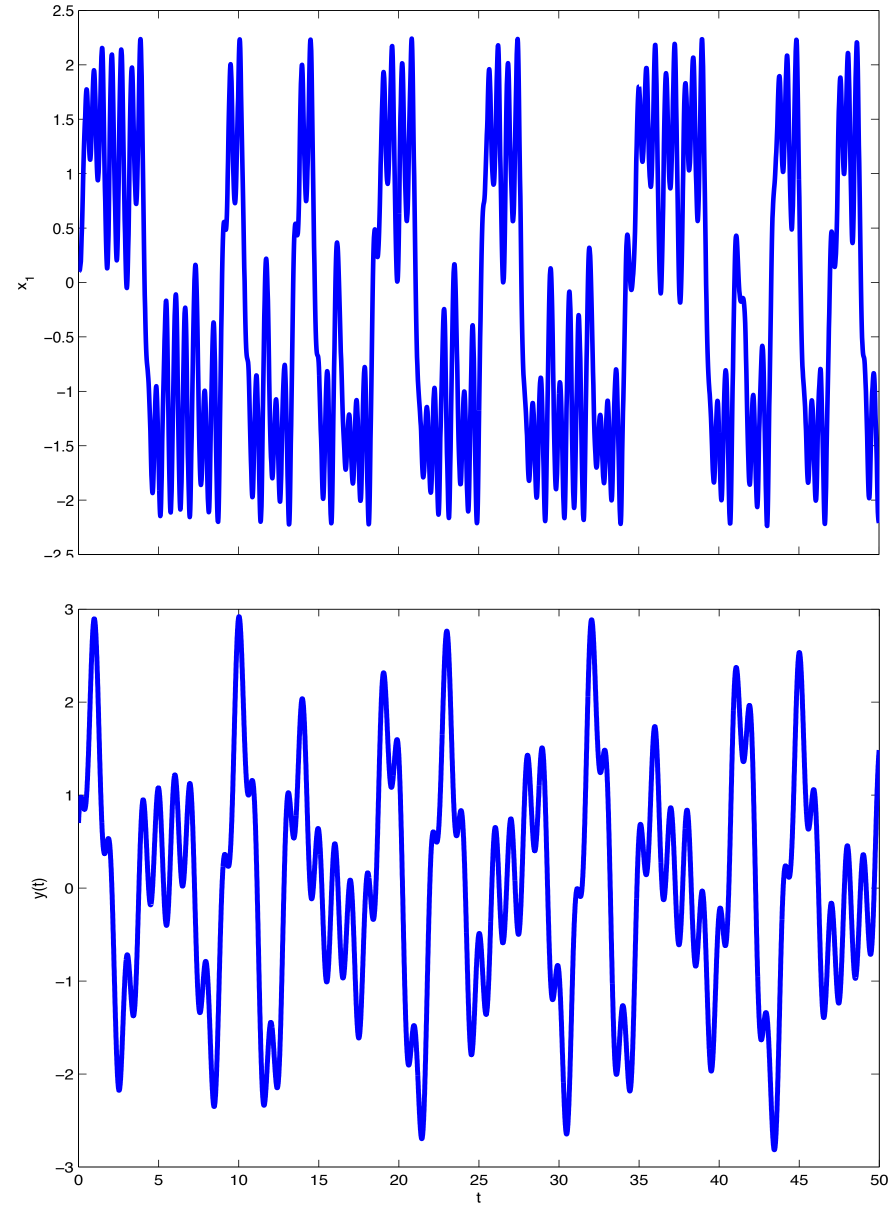
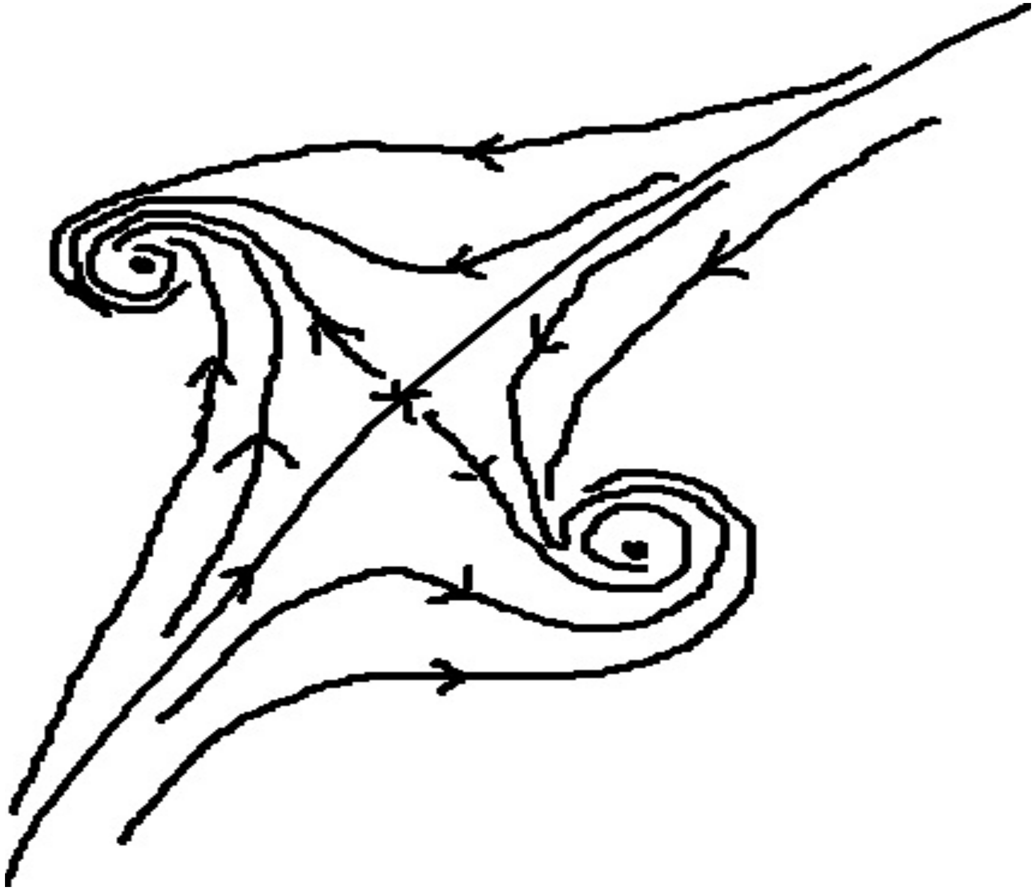
$$\dot{x}_1 = 2x_2$$
$$\dot{x}_2 = -2x_1$$
$$\dot{x}_3 = 2\pi x_4$$
$$\dot{x}_4 = -2\pi x_3$$
$$\dot{x}_5 = \sqrt{2}x_6$$
$$\dot{x}_6 = -\sqrt{2}x_5$$
$$y = x_1 + x_3 + x_5$$



Property 2: Sensitivity to initial conditions

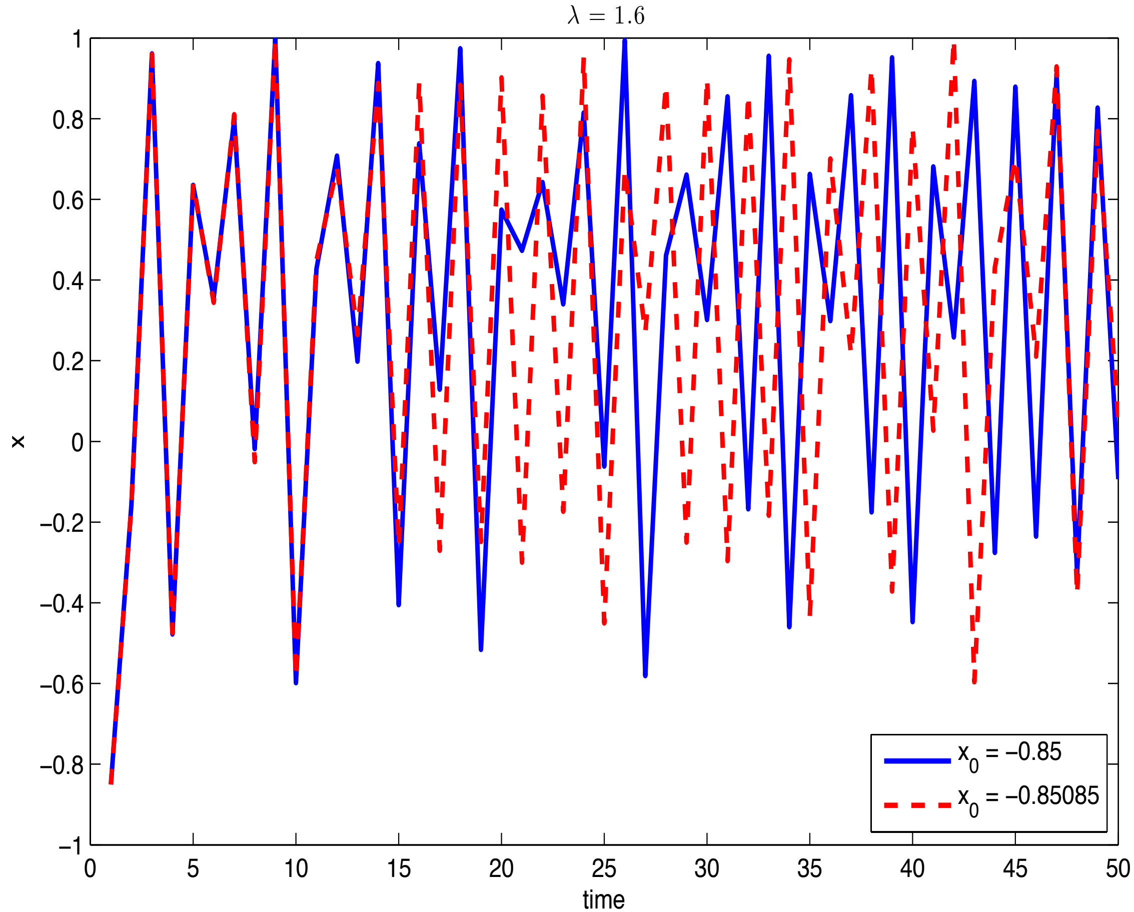
- A feature of nonlinear systems that solutions are bounded and depend on initial conditions.
- But they do not always repel each other.
- In a chaotic system, yes.



Property 2: Sensitivity to initial conditions

$$x(t+1) = 1 - \lambda x^2(t)$$

- ❑ $\lambda = 1.6$
- ❑ $x_0 = -0.85$
- ❑ $x_0 = -0.85085$



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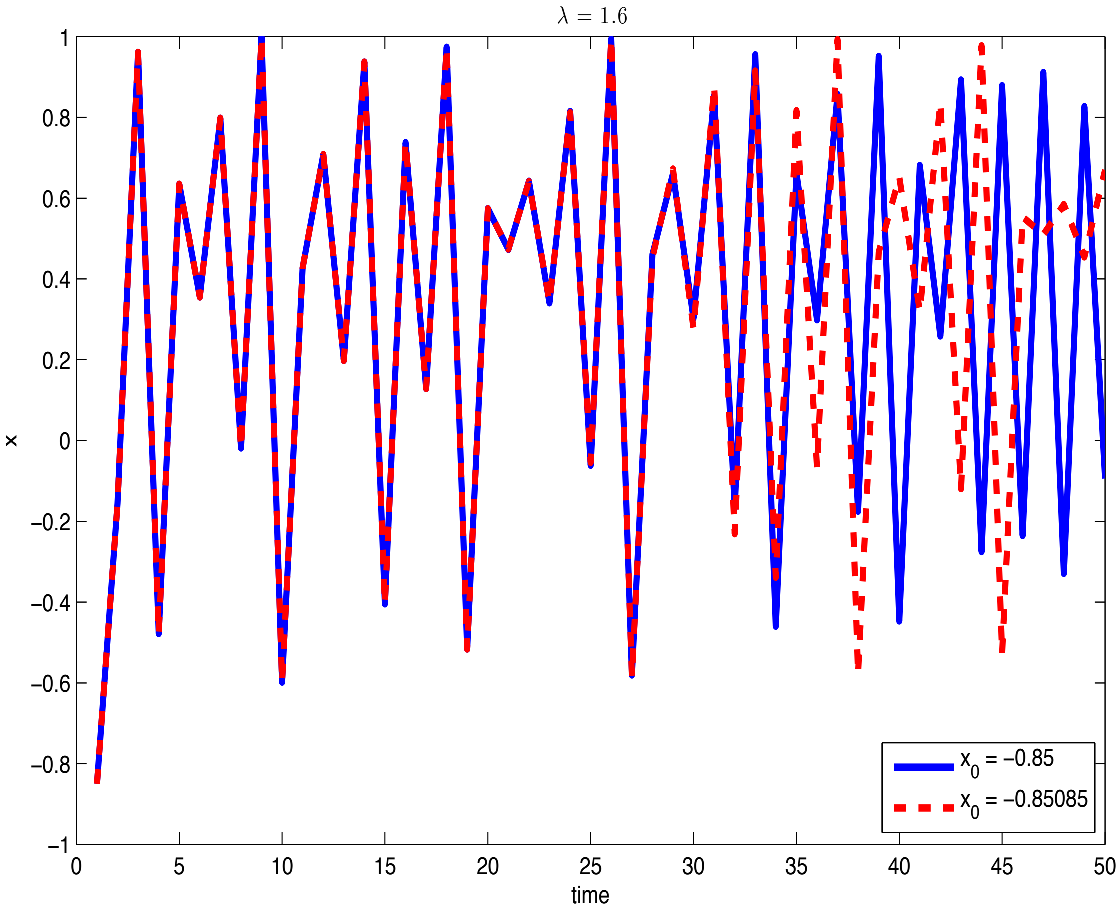


Lyapunov Exponents for 1-dim Maps

- $x(t+1) = F\big(x(t)\big)$ with $F: \mathbb{R} \to \mathbb{R}$ continuously differentiable (C^1)
- Let $x(t)$, $\tilde{x}(t)$ be two solutions s. t. $|\Delta x(0)| = |\tilde{x}(0) - x(0)|$ is small.
- Then $\Delta x(t) = \tilde{x}(t) - x(t) = \Phi\big(t, \tilde{x}(0)\big) - \Phi\big(t, x(0)\big)$ evolves approx. as

$$\Delta x(t) = M(t)\Delta x(0)$$

  with $M(t) = \frac{\partial \Phi}{\partial x_0}\big(t, x(0)\big)$.
- Variational Equation: $M(t+1) = \frac{\partial F}{\partial x}\big(x(t)\big)M(t)$
- Therefore $M(t) = \frac{\partial F}{\partial x}\big(x(t-1)\big)\frac{\partial F}{\partial x}\big(x(t-2)\big)\cdots\frac{\partial F}{\partial x}\big(x(1)\big)\frac{\partial F}{\partial x}\big(x(0)\big)M(0)$

- Let us set $\alpha(t)$ such that $\frac{|\Delta x(t)|}{|\Delta x(0)|} = e^{\alpha(t)\cdot t}$. Then

$$\alpha(t) = \frac{1}{t}\ln\frac{|\Delta x(t)|}{|\Delta x(0)|} \approx \frac{1}{t}\sum_{\tau=0}^{t-1}\ln\left|\frac{\partial F}{\partial x}\big(x(\tau)\big)\right|$$

Lyapunov Exponents for 1-dim Maps

- $x(t+1) = F\big(x(t)\big)$ with $F: \mathbb{R} \to \mathbb{R}$ continuously differentiable (C^1)
- Let us set $\alpha(t)$ such that $\frac{|\Delta x(t)|}{|\Delta x(0)|} = e^{\alpha(t)\cdot t}$. Then

$$\alpha(t) = \frac{1}{t} \ln \frac{|\Delta x(t)|}{|\Delta x(0)|} \approx \frac{1}{t} \sum_{\tau=0}^{t-1} \ln \left| \frac{\partial F}{\partial x}\big(x(\tau)\big) \right|$$

- $\alpha(t)$ is the time-average exponential speed of growth or contraction in [0,*t*] along solution $x(t)$. Can be computed along any solution $x(t)$.
- If the limit exists,

$$\alpha = \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \ln \left| \frac{\partial F}{\partial x}\big(x(\tau)\big) \right|$$

  is the Lyapunov exponent of solution $x(t)$.
- If $\alpha < 0$, the solution $x(t)$ is asymptotically stable.
- If $\alpha > 0$, the solution $x(t)$ is unstable.
- If all solutions are bounded and (almost all) have a positive Lyapunov exponent $\alpha > 0$, then the system is chaotic.

Lyapunov Exponents: Examples

- If the limit exists, the Lyapunov exponent of solution $x(t)$ is

$$\alpha = \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \ln \left| \frac{\partial F}{\partial x}\big(x(\tau)\big) \right|$$

- If $x(t)$ converges to a fixed point $\bar{x}$,

$$\alpha = \ln \left| \frac{\partial F}{\partial x}(\bar{x}) \right|$$

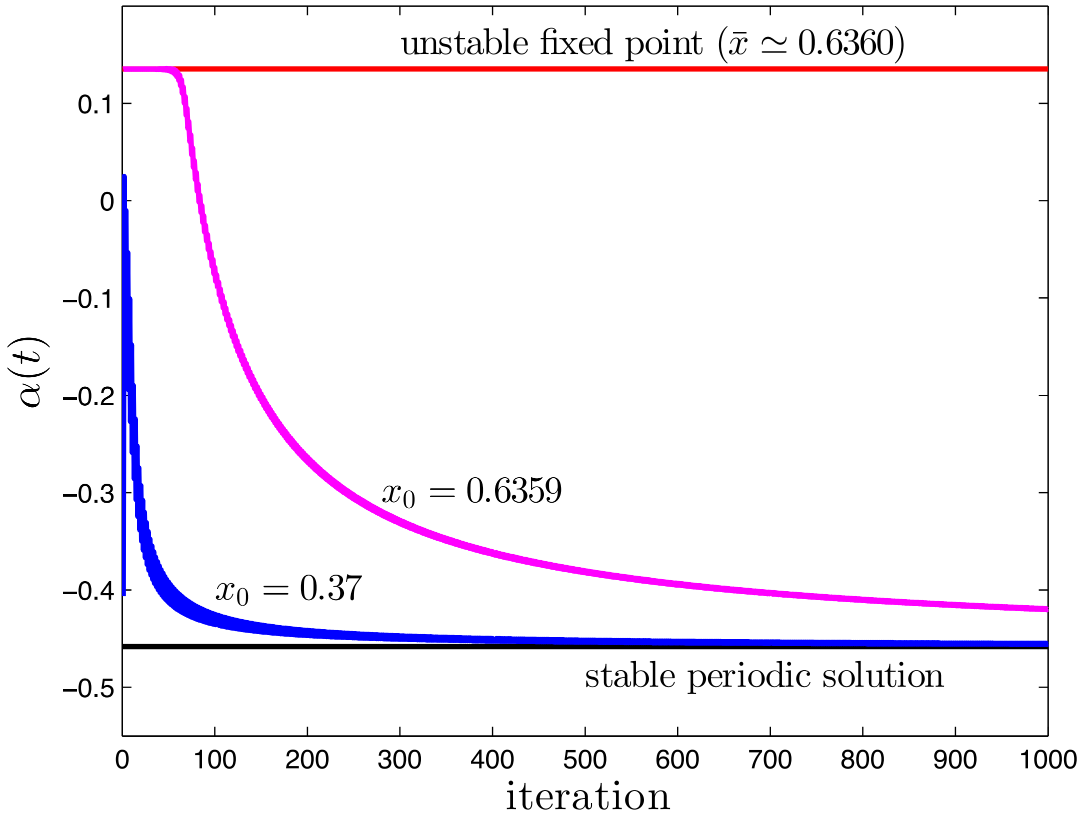
- If $x(t)$ converges to a T-periodic solution $\xi = (\xi_1, \xi_2, \cdots, \xi_T)$,

$$\alpha = \frac{1}{T} \sum_{i=1}^{T} \ln \left| \frac{\partial F}{\partial x}(\xi_i) \right|$$

Example: Lyapunov Exponent of Logistic Map

$$x(t+1) = 1 - \lambda x^2(t)$$

- $\lambda = 0.9$



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$$\alpha = \frac{1}{T} \sum_{i=1}^{T} \ln \left| \frac{\partial F}{\partial x}(\xi_i) \right|$$

- Does not add anything compared to computing the Jacobian.
- If the system is chaotic, how to compute α?
  - Predict long term behavior of $x(t)$ from deterministically chosen $x(0)$? Difficult, even impossible in a chaotic system when initial data have limited precision.
  - Predict long term behavior of $x(t)$ from randomly chosen $x(0)$? Use tools from ergodic theory.

Elements from the Theory of Ergodic Dynamical Systems.

- Probability Space (Ω,Σ,P)
  - Sample space Ω
  - Sigma-algebra Σ
  - Probability measure P
- Transformations F: $\Omega \rightarrow \Omega$
  - Measurable
  - Measure-preserving
  - Invariant set
- Ergodic Transformations F: $\Omega \rightarrow \Omega$
  - Ergodic transformation
  - Mixing transformations
  - Birkhoff Ergodic Theorem

Probability Space (Ω,Σ,P)

- Sample space Ω = {elementary events}
- σ–algebra Σ = {events Σ_i} = {subsets of Ω, including $\emptyset$ and Ω}
  - closed under Complement: If it contains Σ_i it also contains $\Sigma_i^c = \Omega \backslash \Sigma_i$
  - closed under countable (finite or infinite) union of events.
- Probability measure P: $\Sigma \rightarrow [0,1]$, with
  - $P(\emptyset) = 0$ and
  - $P(\cup_i \Sigma_i) = \sum_i P(\Sigma_i)$ for countable (finite or infinite) seq. of disjoint sets $\Sigma_i \in \Sigma$.
- If Ω is a countable set,
  - Largest σ–algebra Σ = power set of all its subsets (default σ–algebra)
  - Ex: Ω = {a,b,c} $\Longrightarrow$ Σ = {$\emptyset$,{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}}
  - Smallest σ–algebra Σ = {$\emptyset$, Ω}
  - Ex: Ω = {a,b,c} $\Longrightarrow$ Σ = {$\emptyset$,{a,b,c}}
  - If A is a (collection of) subset(s) of Ω, σ–algebra generated by A = Smallest σ–algebra that includes A.
  - Ex: Ω = {a,b,c} and A = {a} $\Longrightarrow$ Σ = {$\emptyset$,{a},{b,c},{a,b,c}}.
- If Ω is a non-countable set, need to specify σ–algebra Σ.
  - Ex: $\Omega = \mathbb{R} \Longrightarrow$ σ–algebra generated by all open intervals in $\mathbb{R}$.

Probability Space (Ω,Σ,P): Examples

❑ Example 1: Single dice roll

- Sample space $\Omega = \{1,2,3,4,5,6\}$
- σ–algebra Σ = power set of all its subsets = $\{\emptyset,\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{1,2\},\ldots,\{2,3,4,5,6\},\{1,2,3,4,5,6\}\}$
- Probability measure $P : \Sigma \rightarrow [0,1]$
  $P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = 1/6$.

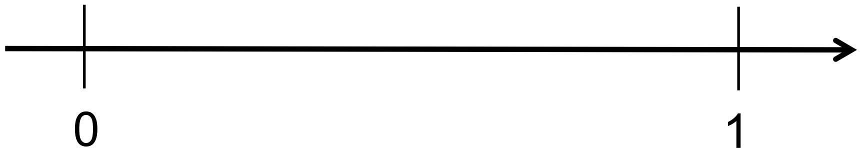
❑ Example 2: Repeated dice roll

- Sample space $\Omega = \{\omega \mid \omega = (\omega_1, \omega_2, \omega_3,\ldots)\}$ with $\omega_i \in \{1,2,3,4,5,6\}$.
- ”cylinder” σ–algebra Σ generated by cylinder subsets
  $S_{(i_1,j_1)(i_2,j_2)\ldots(i_n,j_n)} = \{\omega | \omega_{i_1} = j_1, \omega_{i_2} = j_2, \ldots, \omega_{i_n} = j_n\}$ for some finite $n \in \mathbb{N}^*$, $i_1 < i_2 < \cdots < i_n \in \mathbb{N}^*$ and $j_1, j_2,\ldots, j_n \in \{1,2,3,4,5,6\}$.
- Probability measure $P : \Sigma \rightarrow [0,1]$
  $P(\{S_{(i_1,j_1)(i_2,j_2)\ldots(i_n,j_n)}\}) = (1/6)^n$.

Probability Space (Ω,Σ,P): Examples

- Example 3: Uniform distribution on [0,1)
  - Sample space $\Omega = [0,1)$
  - Borel σ–algebra Σ of [0,1) = smallest σ–algebra containing [a,b), (a,b), (a,b], [a,b] for all $0 \le a \le b \le 1$.
  - Probability measure P = Lebesgue measure $P : \Sigma \to [0,1]$

    P([a,b)) = P((a,b)) = P((a,b]) = P([a,b]) = b-a.



Measurable and Measure-Preserving Transformations

- ❑ Probability space (Ω, Σ, P)
- ❑ Transformation $F: \Omega \rightarrow \Omega$ is
  - Measurable if $F^{-1}(A) \in \Sigma$ for all $A \in \Sigma$.
  - Measure preserving if F is measurable and $P(F^{-1}(A)) = P(A)$ for all $A \in \Sigma$. P is then an invariant measure under F.
- ❑ $A \in \Sigma$ is an invariant set under F if $F^{-1}(A) = A$.
- ❑ Example 1 (continued) : F = Permutation
  - Measure preserving
  - Invariant sets:

ω	1	2	3	4	5	6
$F(\omega)$	2	1	4	6	3	5

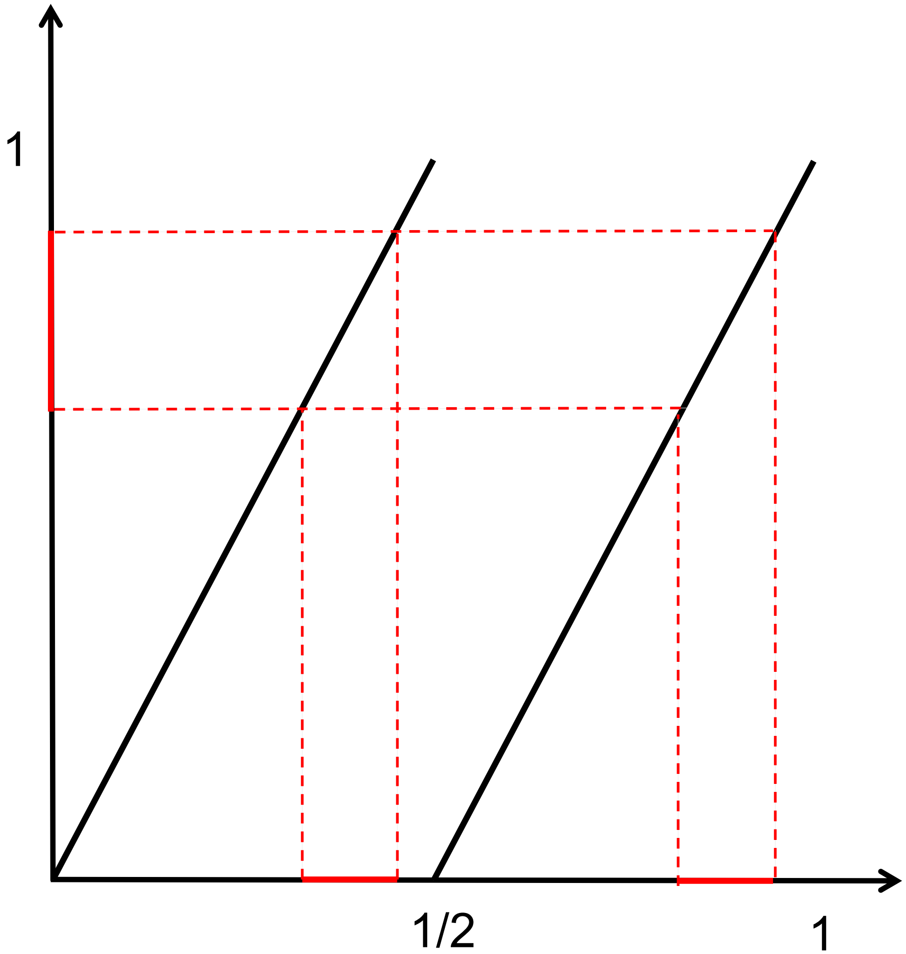
ω	1	2	3	4	5	6
$F(\omega)$	6	1	2	3	4	5

Measurable and Measure-Preserving Transformations

- Example 2 (continued) : F = Left shift transformation
  - $F(\omega_1, \omega_2, \omega_3, \ldots) = (\omega_2, \omega_3, \omega_4, \ldots)$
  - "cylinder" σ–algebra Σ generated by cylinder subsets

    $S_{(i_1,j_1)(i_2,j_2)\ldots(i_n,j_n)} = \{\omega | \omega_{i_1} = j_1, \omega_{i_2} = j_2, \ldots, \omega_{i_n} = j_n\}$ for some finite n $\in \mathbb{N}^*$, $i_1 < i_2 < \cdots < i_n \in \mathbb{N}^*$ and $j_1, j_2, \ldots, j_n \in \{1,2,3,4,5,6\}$.
  - $F^{-1}(S_{(i_1,j_1)(i_2,j_2)\ldots(i_n,j_n)}) = S_{(i_1+1,j_1)(i_2+1,j_2)\ldots(i_n+1,j_n)}$
  - Measure preserving
  - Set $S_{(i_1,j_1)(i_2,j_2)\ldots(i_n,j_n)}$ is not invariant.

Measurable and Measure-Preserving Transformations



- Example 3 (continued) : F = Bernouilli map on $\Omega = [0,1)$
  - F(x) = 2x mod 1.
  - $F^{-1}([a,b)) = [a/2,b/2) \cup [(a+1)/2,(b+1)/2)$
  - Measure preserving
  - [a,b) is not invariant if 0 < a < b < 1.
  - [0,1) is invariant
  - $\mathbb{Q} \cap [0,1)$ is invariant

Ergodic Transformations

- Probability space (Ω, Σ, P)
- Let $F: \Omega \rightarrow \Omega$ be a measure preserving transformation.
- F is ergodic if for any set $A \in \Sigma$ that is invariant under F,

  $P(A) = 0$ or $P(A) = 1$.
- Then the invariant measure P is an ergodic measure w.r.t. F.
- F is **not** ergodic w.r.t. P $\Leftrightarrow$ there is a non trivial $\Omega' \subset \Omega$, $P_1 \neq P_2$ both invariant under F and such that $P_1(\Omega') = 1$, $P_2(\Omega') = 0$ and some $0 < \lambda < 1$ with $P = \lambda P_1 + (1- \lambda)P_1$.
- F is ergodic w.r.t. invariant P if its only invariant sets are Ω and $\emptyset$.
- F is mixing if for any two sets $A, B \in \Sigma$,

  $\lim_{N\rightarrow\infty} P(A \cap F^{-(N)}(B)) = P(A)P(B)$.
- F is mixing $\Rightarrow$ F is ergodic (proof : take B = A an invariant set under F).

Ergodic Transformations

- Example 1 (continued) : F = Permutation

ω	1	2	3	4	5	6
$F(\omega)$	2	1	4	6	3	5

- $P(\{\omega\}) = 1/6$ if $\omega = 1, 2, 3, 4, 5, 6$.
- F is **not** ergodic for P $\Leftrightarrow$ there is a non trivial $\Omega' \subset \Omega$, $P_1 \neq P_2$ both invariant under F and such that $P_1(\Omega') = 1$, $P_2(\Omega') = 0$ and some $0 < \lambda < 1$ with $P = \lambda P_1 + (1-\lambda)P_2$
- $P_1(\{\omega\}) = 1/2$ if $\omega = 1, 2$ and 0 otherwise invariant under F
- $P_2(\{\omega\}) = 1/4$ if $\omega = 3, 4, 5, 6$ and 0 otherwise invariant under F
- $\Omega' = \{1,2\}$ non trivial $\Rightarrow$ $P_1(\Omega') = 1$ and $P_2(\Omega') = 0$
- $P = \lambda P_1 + (1-\lambda)P_2$ for $\lambda = 1/3$
- Therefore F is not ergodic.

Ergodic Transformations

- Example 1 (continued) : F = Permutation

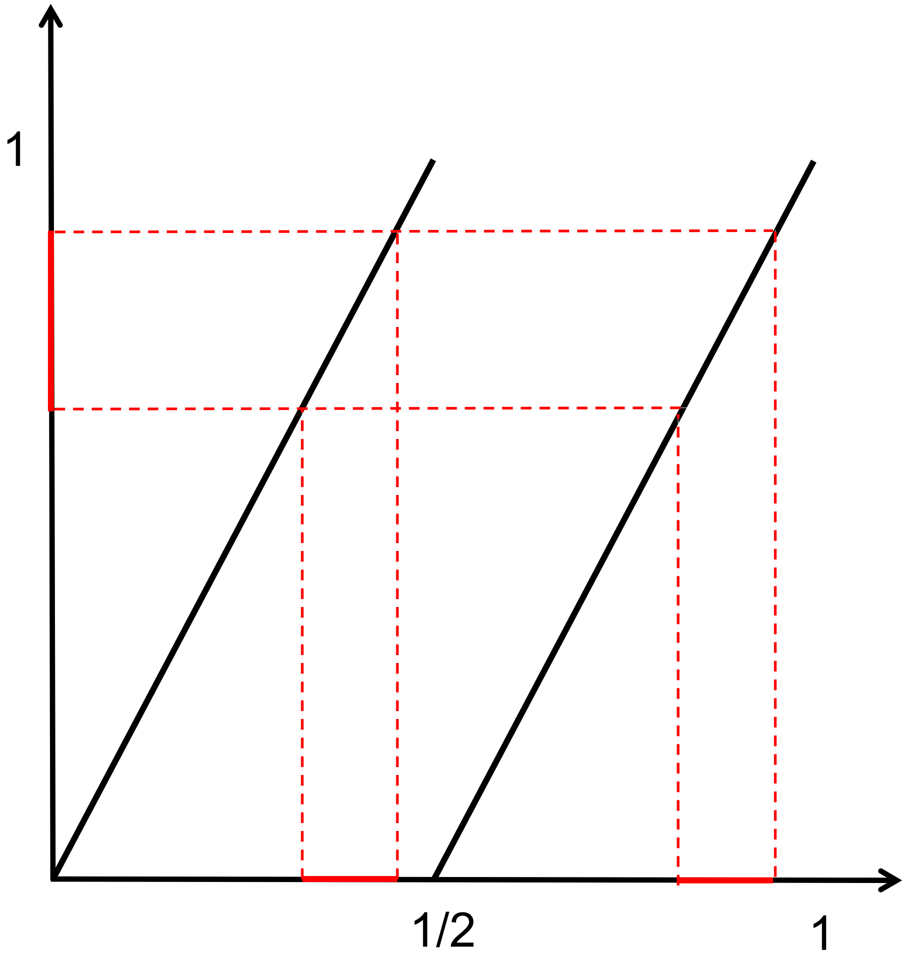
ω	1	2	3	4	5	6
$F(\omega)$	6	1	2	3	4	5

- $P(\{\omega\}) = 1/6$ if $\omega = 1, 2, 3, 4, 5, 6$ invariant under F.
- The only invariant sets are Ω and $\emptyset \Rightarrow$ F is ergodic.
- Mixing?
- Let A = B = {1}. Then $F^{-(N)}(B) = \{i\}$ with i = 1 if and only if N mod 6 = 0.
- Therefore $A \cap F^{-(N)}(B) = \{1\}$ if N mod 6 = 0 and $A \cap F^{-(N)}(B) = \emptyset$ and $\lim_{N\to\infty} P(A \cap F^{-(N)}(B))$ does not exist.
- Therefore F is not mixing.

Ergodic Transformations

- Example 2 (continued) : F = Left shift transformation
  - $F(\omega_1, \omega_2, \omega_3, \ldots) = (\omega_2, \omega_3, \omega_4, \ldots)$
  - "cylinder" σ–algebra Σ generated by cylinder subsets
    $S_{(i_1,j_1)(i_2,j_2)\ldots(i_n,j_n)} = \{\omega | \omega_{i_1} = j_1, \omega_{i_2} = j_2, \ldots, \omega_{i_n} = j_n\}$ for some finite $n \in \mathbb{N}^*$, $i_1 < i_2 < \cdots < i_n \in \mathbb{N}^*$ and $j_1, j_2, \ldots, j_n \in \{1,2,3,4,5,6\}$.
  - $F^{-1}(S_{(i_1,j_1)(i_2,j_2)\ldots(i_n,j_n)}) = S_{(i_1+1,j_1)(i_2+1,j_2)\ldots(i_n+1,j_n)}$
  - Measure preserving
  - Sets $A = S_{(i_1,j_1)(i_2,j_2)\ldots(i_n,j_n)}$ and $B = S_{(k_1,l_1)(k_2,l_2)\ldots(k_m,l_m)}$
  - $F^{-N}(B) = F^{-N}(S_{(k_1,l_1)(k_2,l_2)\ldots(k_m,l_m)}) = S_{(k_1+N,l_1)(k_2+N,l_2)\ldots(k_m+N,l_m)}$
  - Let N be large enough so that $k_1 + N > i_n$
  - Then $A \cap F^{-N}(B) = S_{(i_1,j_1)(i_2,j_2)\ldots(i_n,j_n)(k_1+N,l_1)(k_2+N,l_2)\ldots(k_m+N,l_m)}$
  - $P(A \cap F^{-N}(B)) = (1/6)^{n+m} = (1/6)^n \cdot (1/6)^m = P(A)P(B)$
  - F is mixing for cylinder sets, can show that it is mixing for all sets in Σ.

Ergodic Transformations



- Example 3 (continued) : F = Bernouilli map on $\Omega = [0,1)$
  - F(x) = 2x mod 1.
  - Measure preserving
  - Binary expansion of x = (b_1, b_2, b_3, …)
  - $x = \sum_{i=1}^{\infty} b_i 2^{-i}$
  - $F(x) = 2x \bmod 1 = \sum_{j=1}^{\infty} b_{j+1} 2^{-j} \bmod 1$
  - Binary expansion of F(x) = (b_2, b_3, b_4, …)
  - Left shift

Birkhoff Ergodic Theorem

- Probability space (Ω, Σ, P)
- Random variable f is a function $f : \Omega \to \mathbb{R}$ that is measurable, i.e. such that $f^{-1}(B) \in \Sigma$ for all $B \in \Sigma$ = Borel σ–algebra of $\mathbb{R}$ (note: can take Σ = σ–algebra generated by subsets $(-\infty,b]$ for all $b \in \mathbb{R}$).
- Let $F: \Omega \to \Omega$ be a measure preserving transformation, and P be an invariant measure under F(.).
- Theorem. Let f be a P-integrable random variable (i.e., such that $E[|f|] < \infty$). Then
  - for P-almost all $\omega \in \Omega$ (i.e., except possibly for a set *M* such that P(*M*) = 0)

    $$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f\left(F^{(n)}(\omega)\right) \text{ exists.}$$

  - if in addition F is ergodic, then for P-almost all $\omega \in \Omega$

    $$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f\left(F^{(n)}(\omega)\right) = \int_\Omega f(\omega)dP\left((\omega)\right)$$

Temporal average

Expected value E[f]

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- Probability Space (Ω,Σ,P)
  - Sample space Ω
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  - Measure-preserving
  - Invariant set
- Ergodic Transformations F: $\Omega \rightarrow \Omega$
  - Ergodic transformation
  - Mixing transformations
  - Birkhoff Ergodic Theorem
- Back to Lyapunov Exponents.

Lyapunov Exponents for 1-dim Maps

- $x(t+1) = F\big(x(t)\big)$ with $F\colon \mathbb{R} \to \mathbb{R}$ continuously differentiable (C[1])
- Let us set $\alpha(t)$ such that $\frac{|\Delta x(t)|}{|\Delta x(0)|} = e^{\alpha(t)\cdot t}$. Then

$$\alpha(t) = \frac{1}{t} \ln \frac{|\Delta x(t)|}{|\Delta x(0)|} \approx \frac{1}{t} \sum_{\tau=0}^{t-1} \ln \left| \frac{\partial F}{\partial x} \big(x(\tau)\big) \right|$$

- $\alpha(t)$ is the time-average exponential speed of growth or contraction in [0,*t*] along solution $x(t)$. Can be computed along any solution $x(t)$.
- If the limit exists,

$$\alpha = \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \ln \left| \frac{\partial F}{\partial x} \big(x(\tau)\big) \right|$$

  is the Lyapunov exponent of solution $x(t)$.
- How to compute it?
  - Predict long term behavior of $x(t)$ from determinitstically chosen $x(0)$? Difficult, even impossible in a chaotic system when initial data have limited precision.
  - Predict long term behavior of $x(t)$ from randomly chosen $x(0)$? Use tools from ergodic theory.

Lyapunov Exponents for 1-dim Maps

- If the limit exists, the Lyapunov exponent of solution $x(t)$ is

$$\alpha = \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \ln \left| \frac{\partial F}{\partial x} \big(x(\tau)\big) \right|$$

- Birkhoff's Theorem (i): Let f be a P-integrable random variable. If $F: \Omega \to \Omega$ is a measure preserving transformation (i.e., P is an invariant measure under $F(\cdot)$), then for P-almost all $\omega \in \Omega$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f\big(F^{(n)}(\omega)\big) \text{ exists.}$$

- Pick $\Omega = \mathbb{R}$, $f = \ln\left|{}^{\partial F}/_{\partial x}\right|$ and $\omega = x$.
- Theorem (i): If P is an invariant measure under $F(\cdot)$, then for P-almost all solutions, then the Lyapunov exponent of solution $x(t)$ exists.

Lyapunov Exponents for 1-dim Maps

- If the limit exists, the Lyapunov exponent of solution $x(t)$ is

$$\alpha = \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \ln \left| \frac{\partial F}{\partial x}\big(x(\tau)\big) \right|$$

- Birkhoff's Theorem (ii): Let f be a P-integrable random variable. If $F:\Omega \to \Omega$ is (in addition) an ergodic transformation (i.e., P is an ergodic measure with respect to $F(\cdot)$), then for P-almost all $\omega \in \Omega$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f\big(F^{(n)}(\omega)\big) = \int_\Omega f(\omega) dP\big((\omega)\big)$$

- Pick $\Omega = \mathbb{R}$, $f = \ln\left|{}^{\partial F}/_{\partial x}\right|$ and $\omega = x$.
- Theorem (ii): If P is (in addition) an ergodic measure under $F(\cdot)$, then for P-almost all solutions,

$$\alpha = \int_{-\infty}^{\infty} \ln \left| \frac{\partial F}{\partial x}(x) \right| dP(x)$$

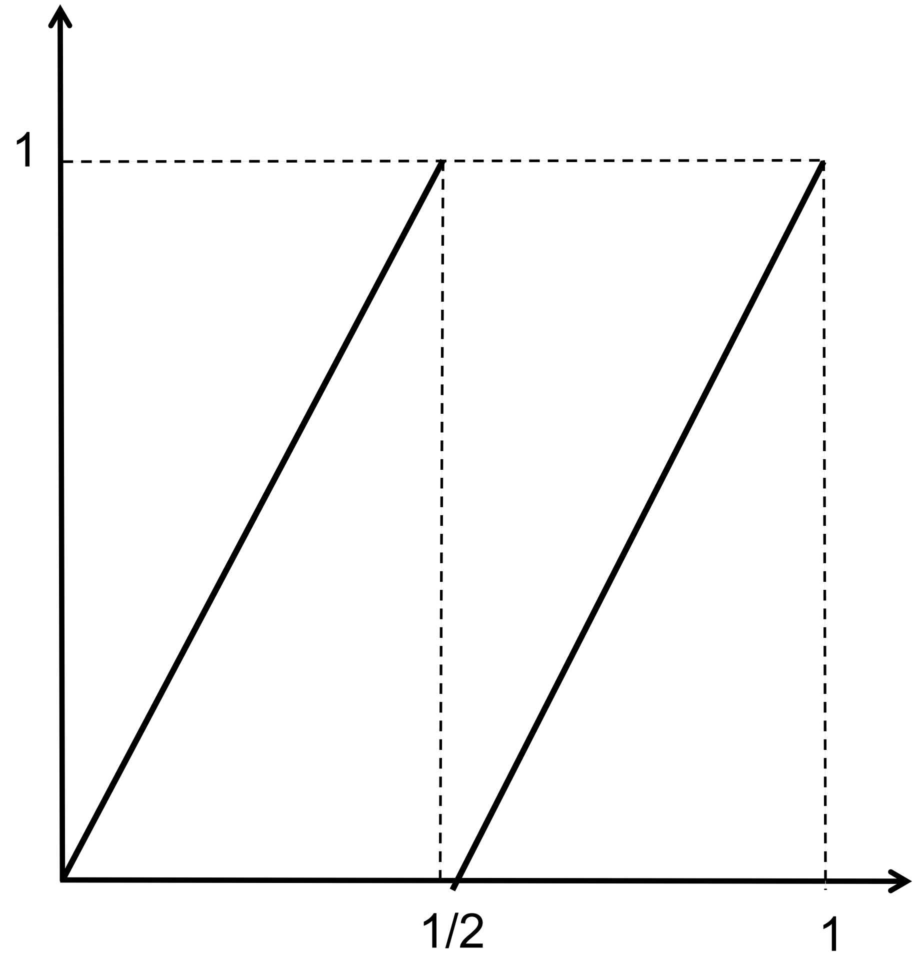
- If the ergodic invariant measure is given by a density $\rho(x)$

$$\alpha = \int_{-\infty}^{\infty} \ln \left| \frac{\partial F}{\partial x}(x) \right| \rho(x) dx$$

Mixing Property of Bernouilli Map

$$x(t+1) = 2x(t) \quad \mathrm{mod}\, 1$$

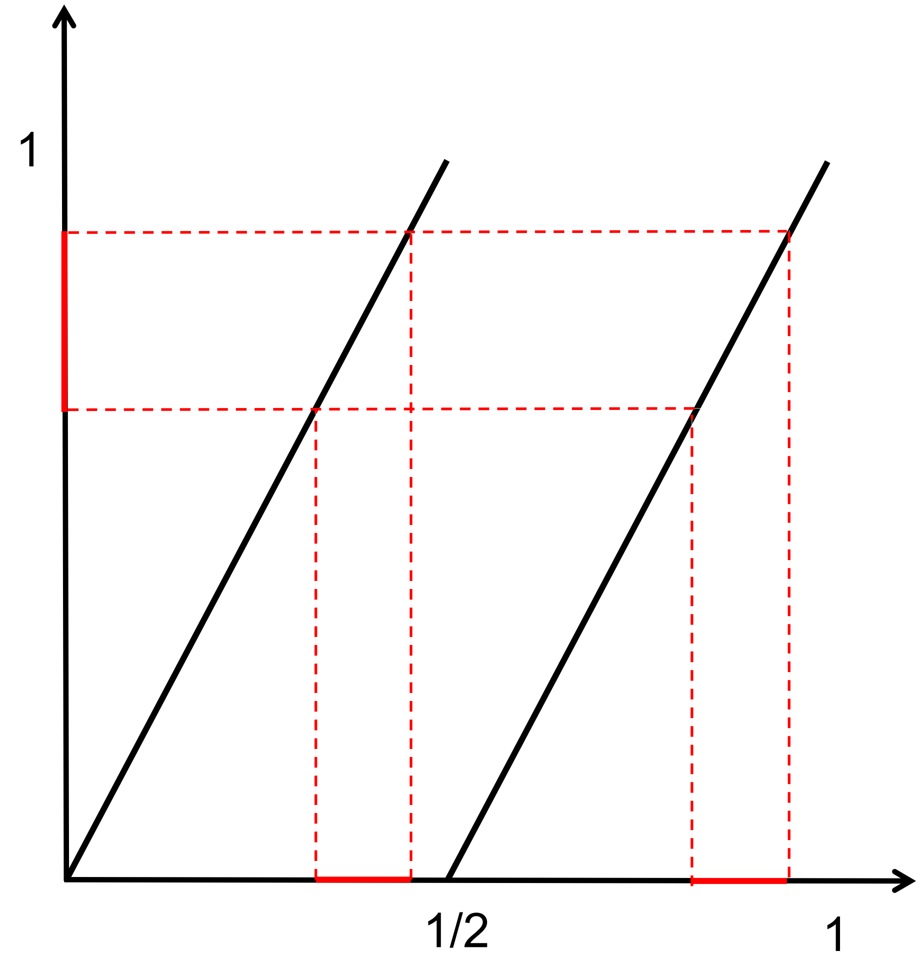
- P([a,b)) = |b-a|.
- P is called the Lebesgue measure.
- $dP(x) = dx$



Mixing Property of Bernouilli Map

$$x(t+1) = 2x(t) \quad \bmod 1$$

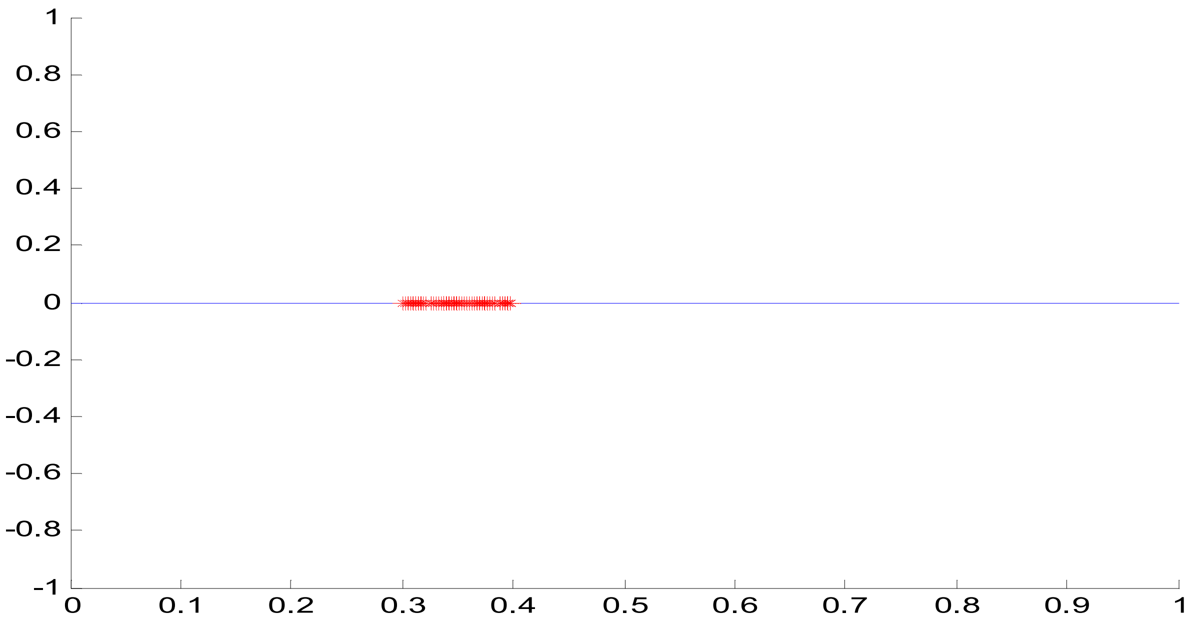
- P([a,b)) = |b-a|.
- P is invariant under F(.).
- $dP(x) = dx$



Mixing Property of Bernouilli Map

$$x(t+1) = 2x(t) \quad \mathrm{mod}\, 1$$

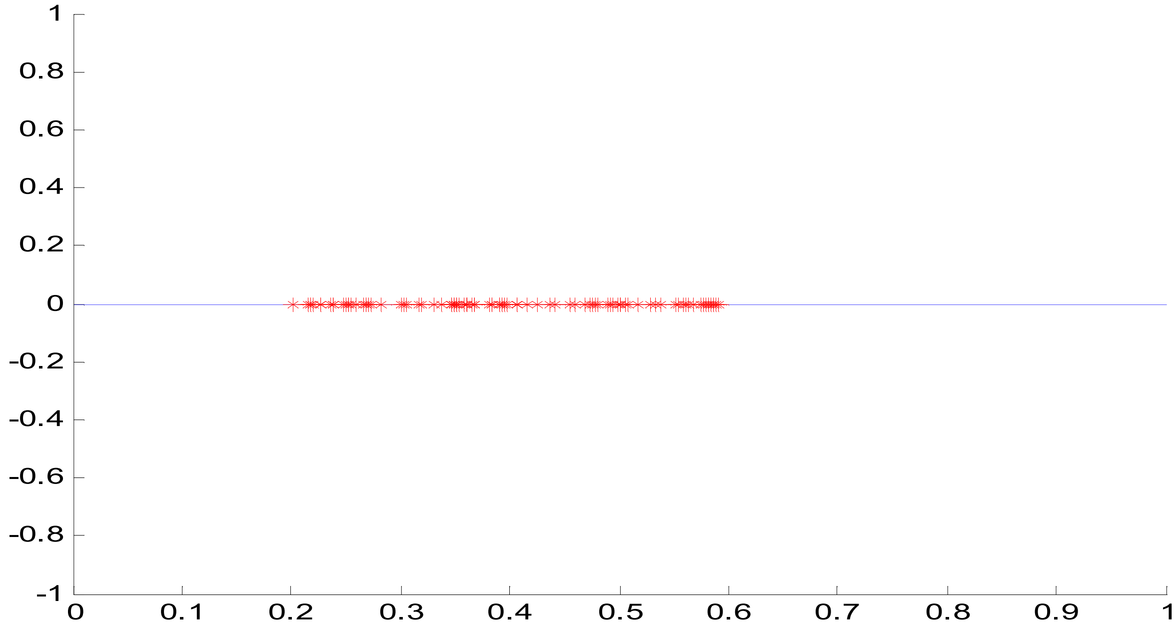
- 100 points chosen randomly in A=[0.3,0.4]



Mixing Property of Bernouilli Map

$$x(t+1) = 2x(t) \quad \mathrm{mod} 1$$

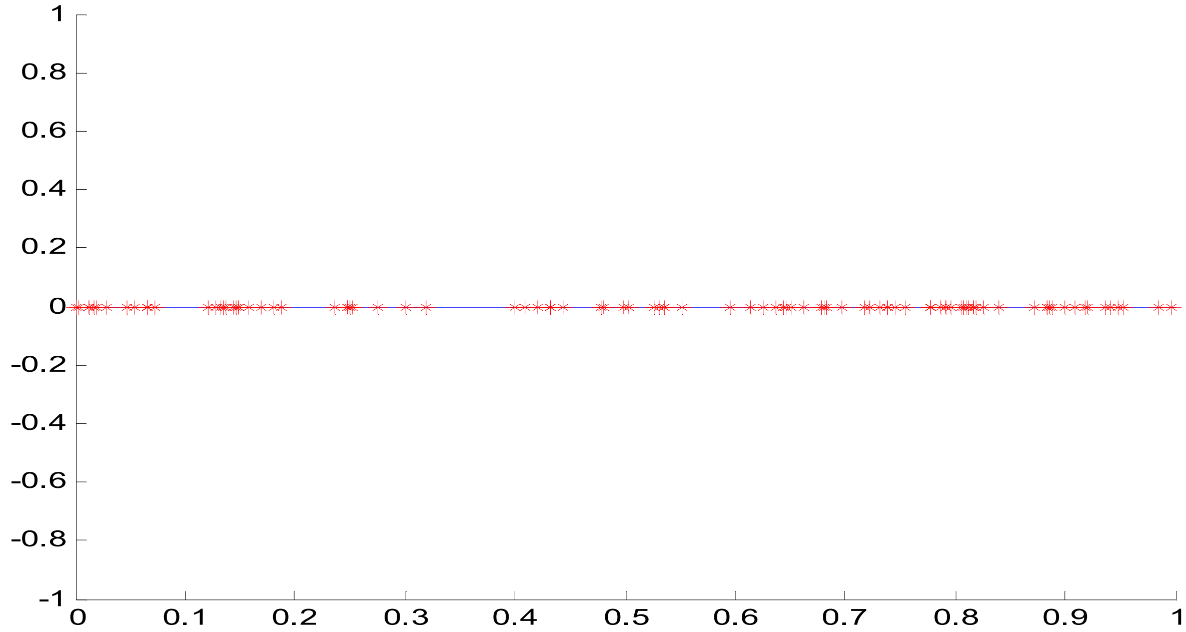
- Position of the 100 points after 2 iterations



Mixing Property of Bernouilli Map

$$x(t+1) = 2x(t) \quad \mod 1$$

- Position of the 100 points after 5 iterations

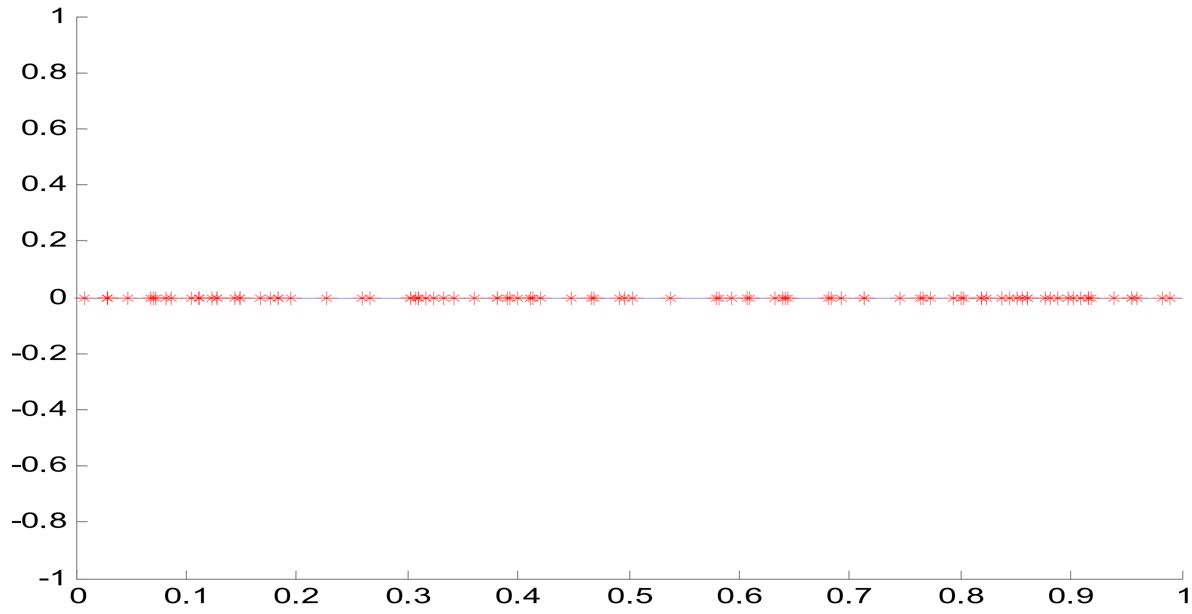


Mixing Property of Bernouilli Map

$$x(t+1) = 2x(t) \quad \mathrm{mod}\,1$$

- Position of the 100 points after 10 iterations
- F is mixing for P

-> P is ergodic with respect to F(.).



Lyapunov Exponents of Bernouilli Map

- Theorem (ii): If P is an ergodic measure under $F(\cdot)$, then for P-almost all solutions,

$$\alpha = \int_{-\infty}^{\infty} \ln \left| \frac{\partial F}{\partial x}(x) \right| dP(x)$$

- Bernouilli map:

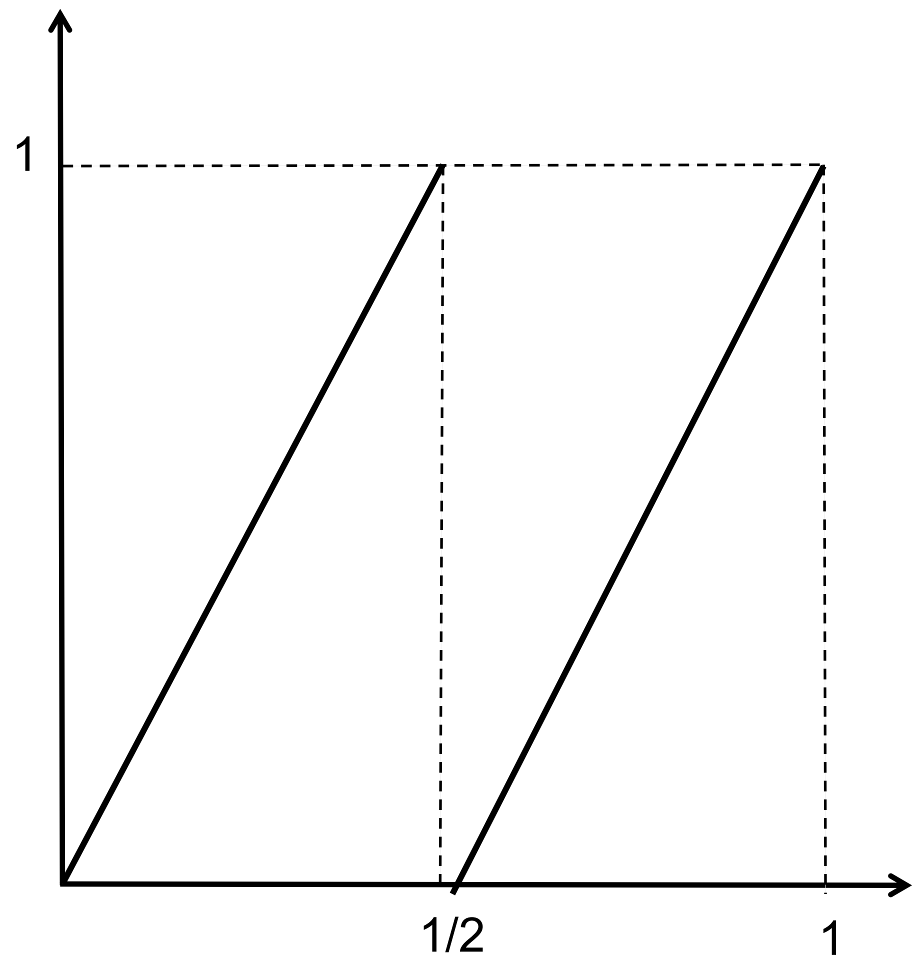
$x(t+1) = 2x(t)$ mod1

  - $\Omega = [0,1)$
  - $\frac{\partial F}{\partial x}(x) = 2$ for all $x \in \Omega \setminus \{^1/_2\}$
  - $dP(x) = dx$

- Therefore

$$\alpha = \int_0^1 \ln 2 \, dx = \ln 2 > 0$$

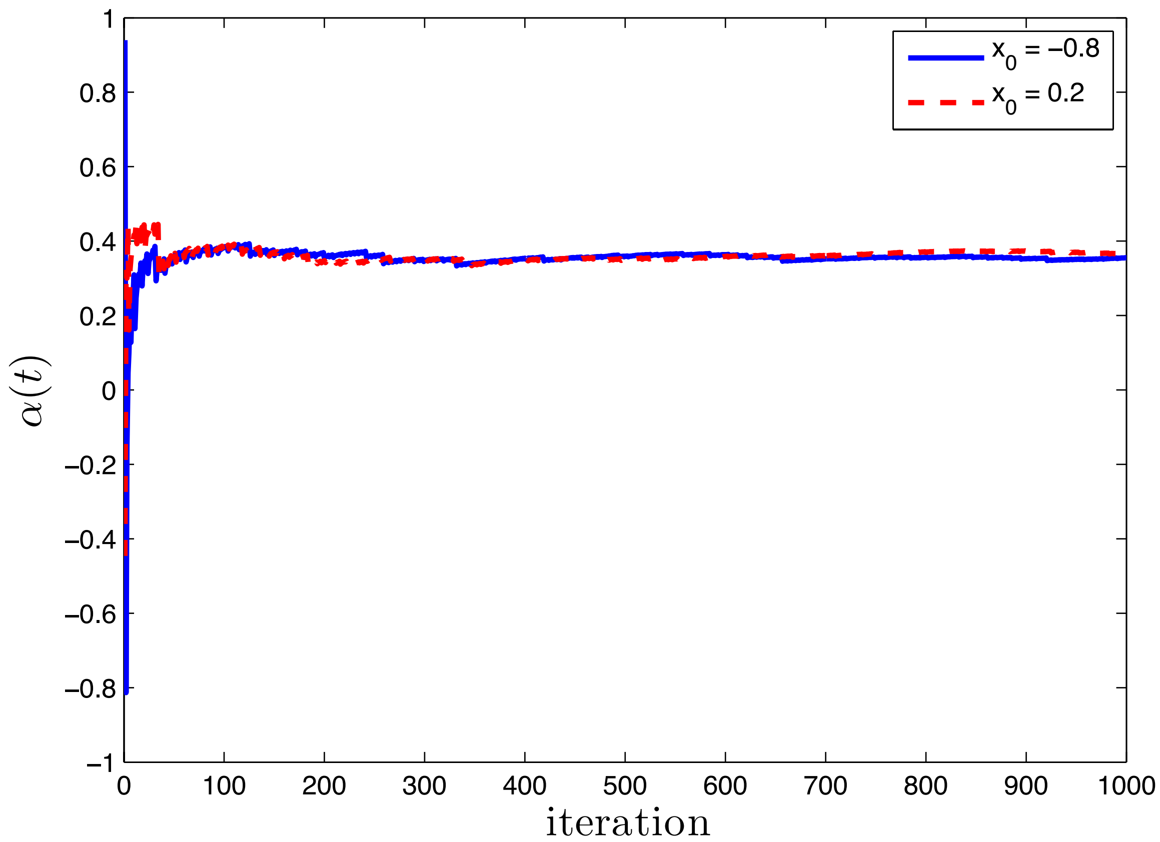
and the system is chaotic.



Lyapunov Exponent of Logistic Map

$$x(t+1) = 1 - \lambda x^2(t)$$

- ❑ $\lambda = 1.6$

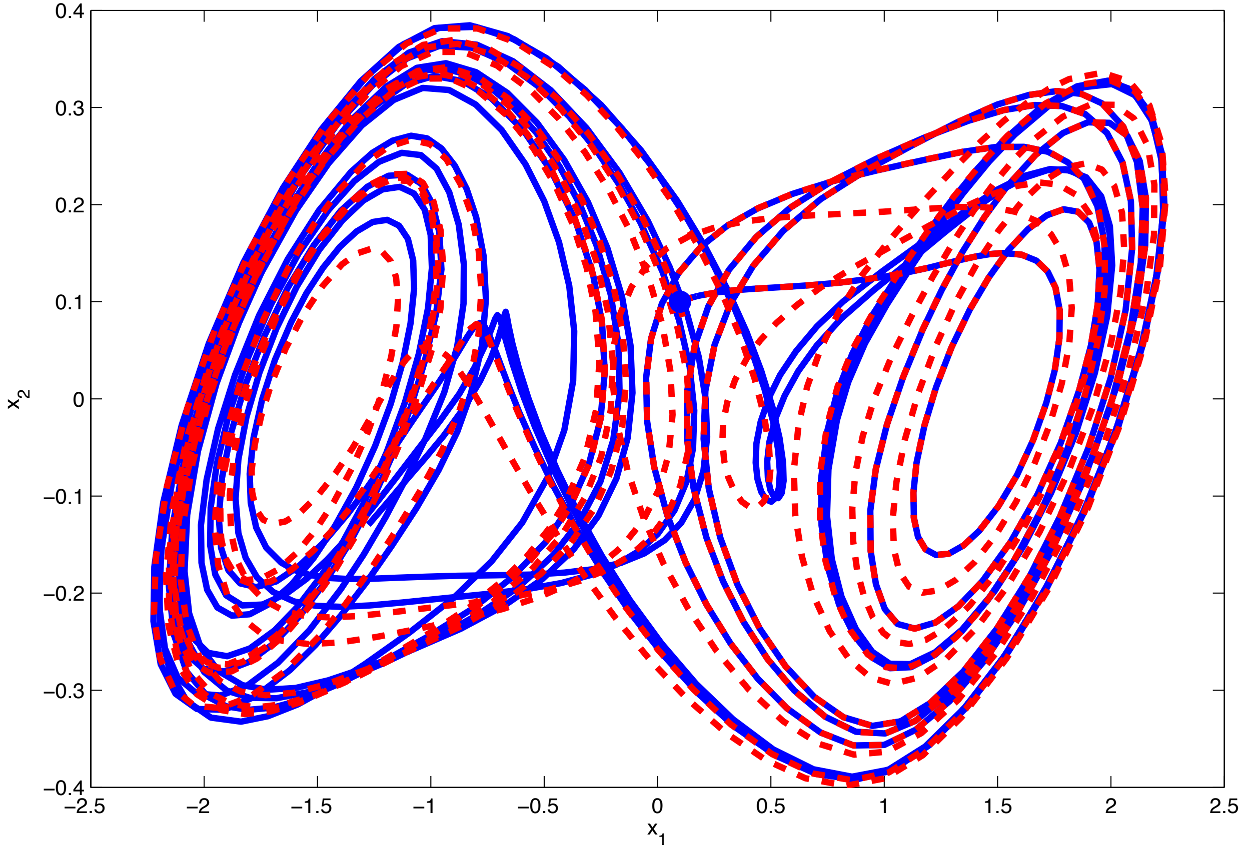


Property 2: Sensitivity to initial conditions

$$\dot{x}_1 = \alpha(-x_1 - f(x_1) + x_2)$$

$$\dot{x}_2 = x_1 - x_2 + x_3$$

$$\dot{x}_3 = -\beta x_2$$



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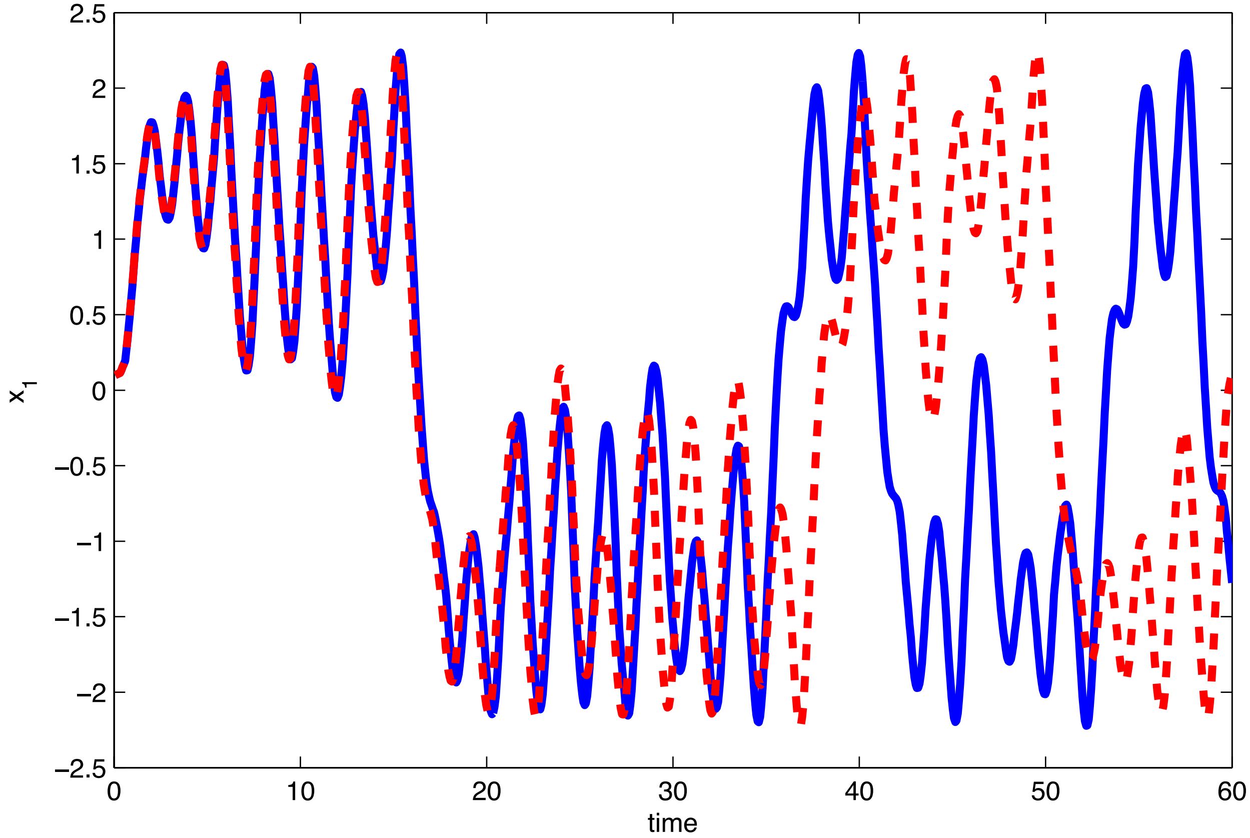
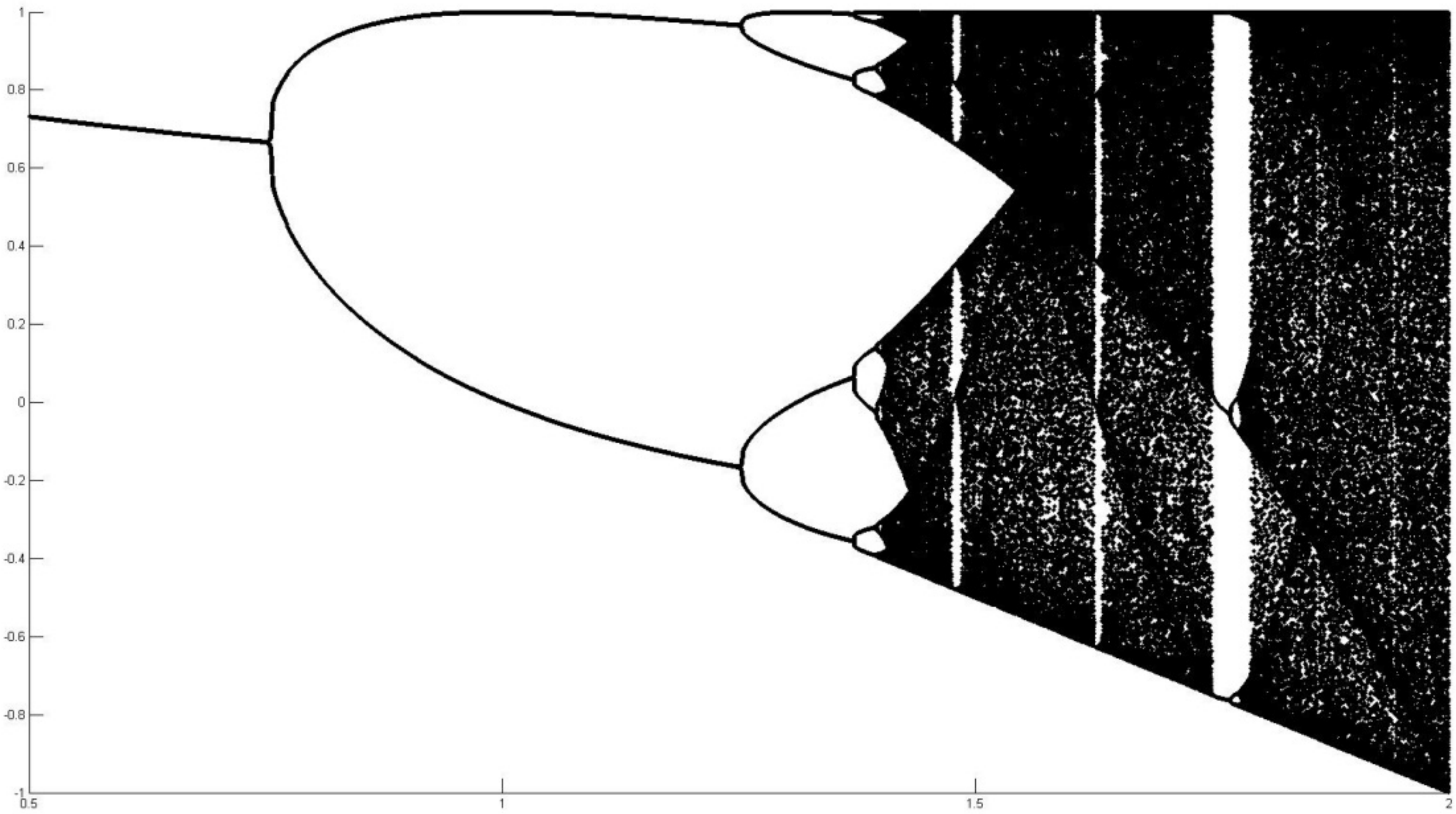


Property 3: Dense set of periodic solutions

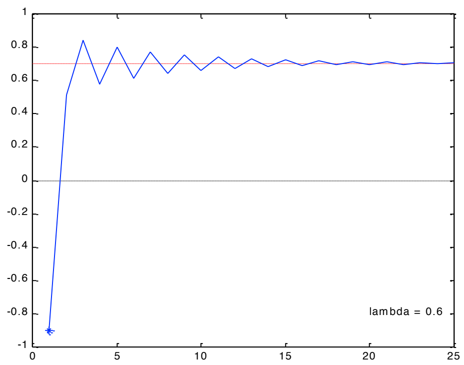
$$x(t+1) = 1 - \lambda x^2(t)$$

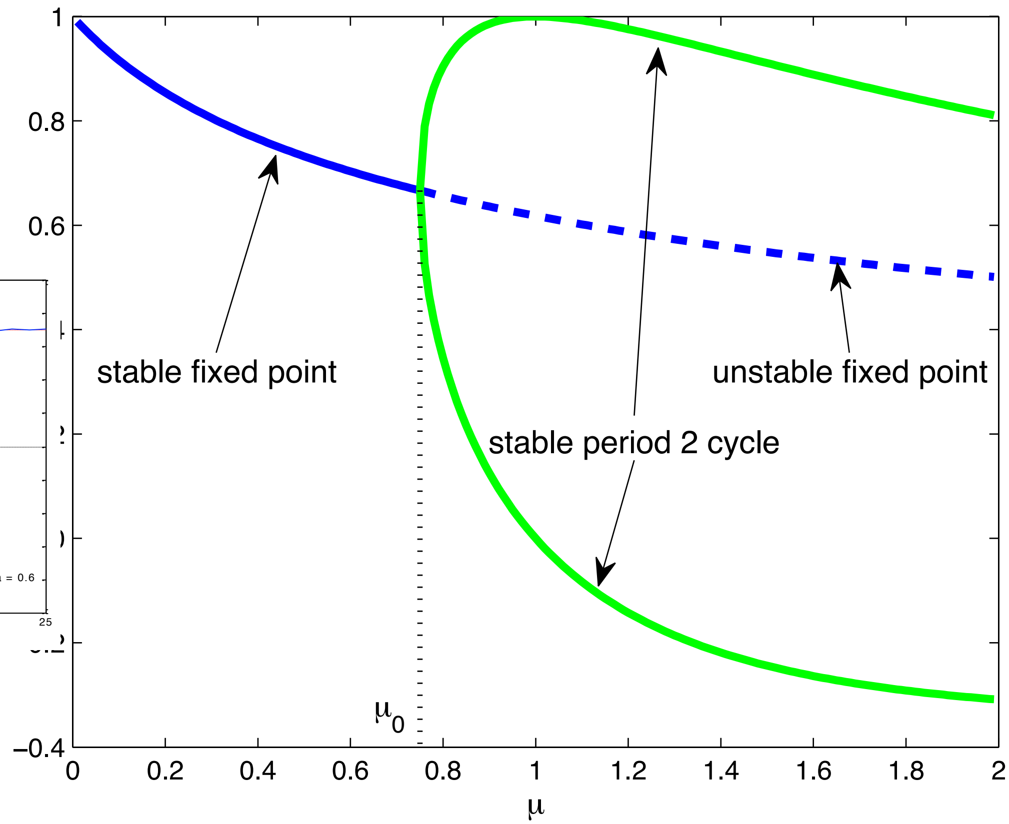


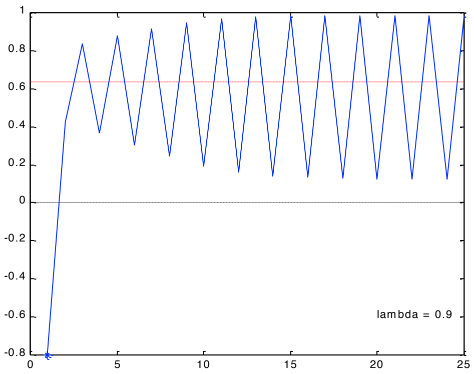
Remember: Flip Bifurcation of Logistic Map

$$x(t+1) = 1 - \lambda x^2(t)$$

Flip bifurcation at $\lambda = \mu_0 = 0.75$

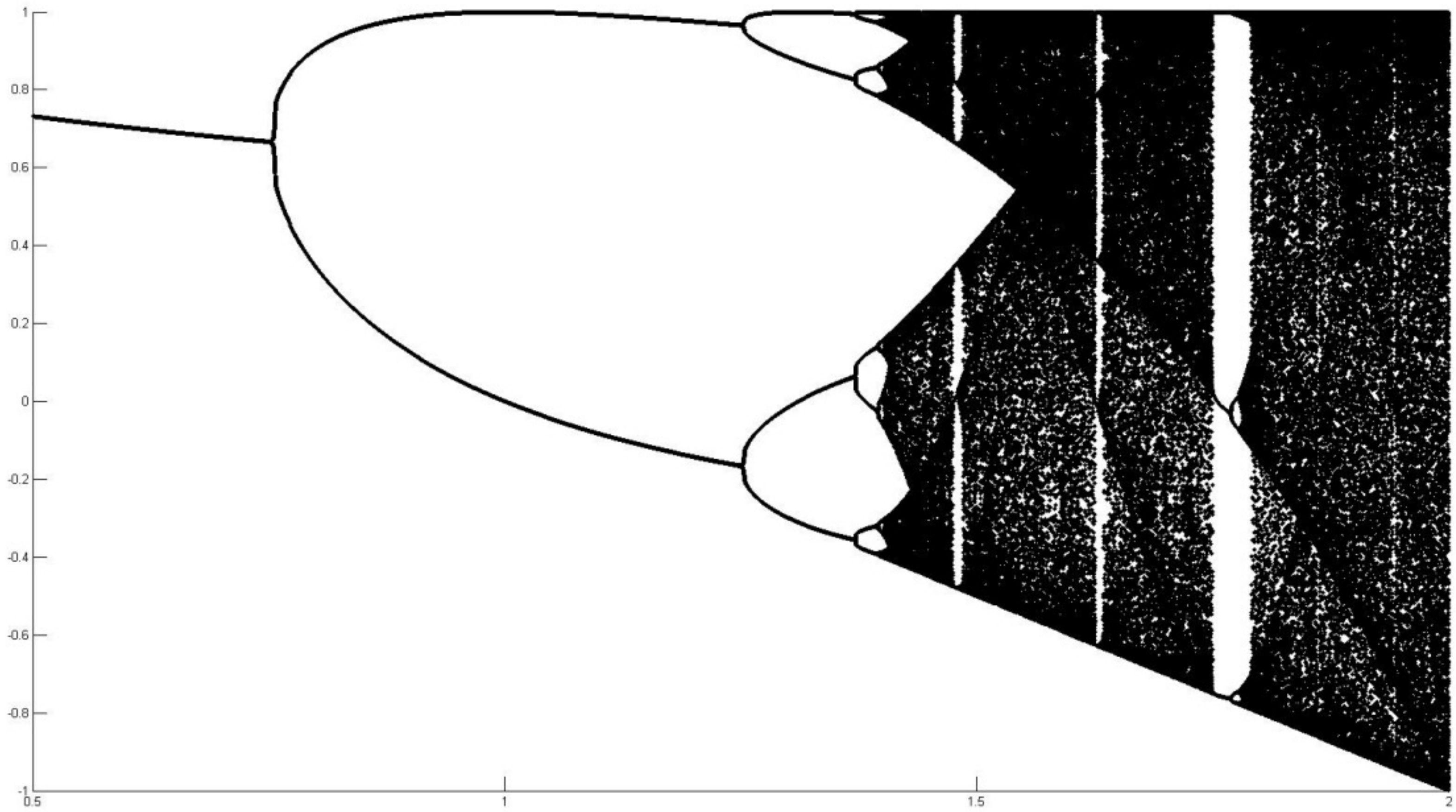






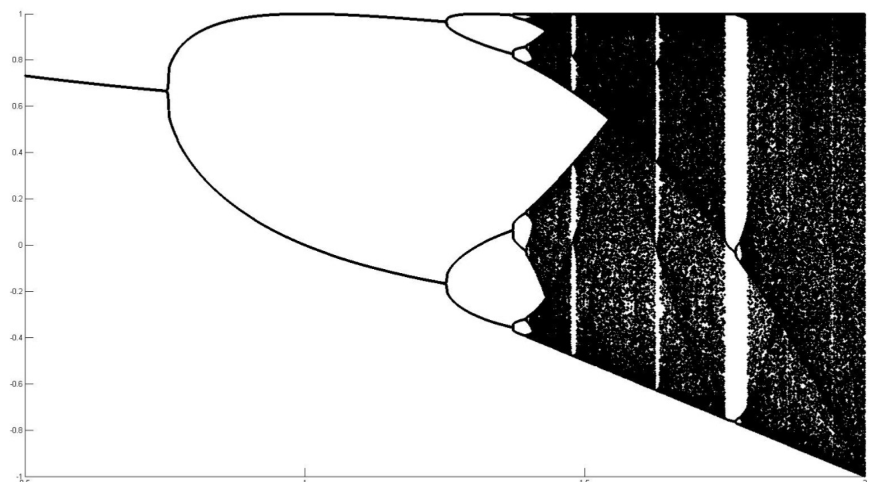
Cascade of Period Doubling Bifurcations

- Period Doubling Bifurcations: $1 \rightarrow 2 \rightarrow 4 \rightarrow 8 \rightarrow \ldots \rightarrow 2^n$
- Ratio of distances between consecutive bifurcation points = 4.669... = Feigenbaum constant.
- Windows in chaos, largest one with a stable 3-periodic solution, followed by new period doubling cascade: $3 \rightarrow 6 \rightarrow 12 \rightarrow 24 \rightarrow \ldots$



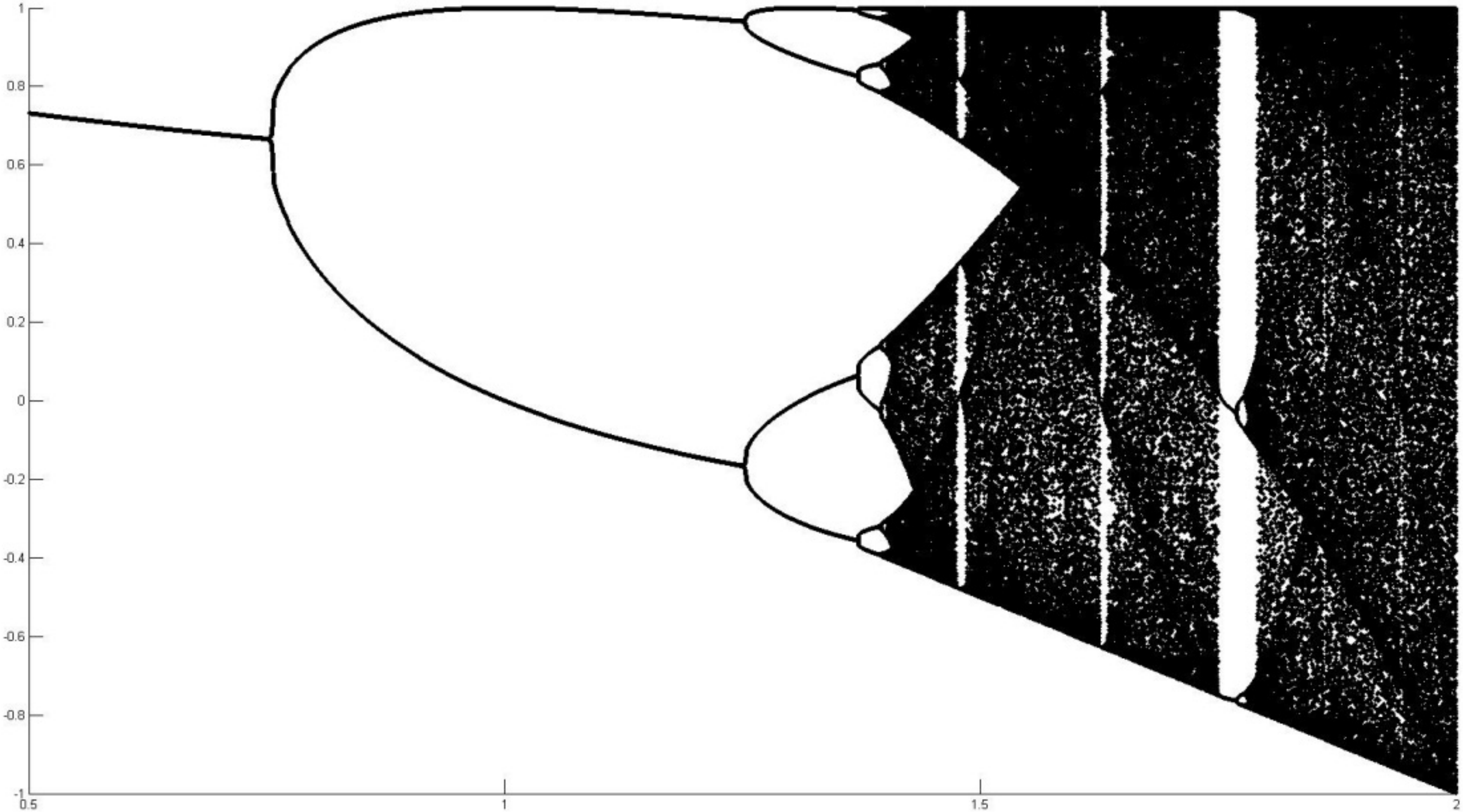
Sarkovskii's Theorem

- Sarkovskii's ordering of natural numbers:
  $3 \triangleright 5 \triangleright 7 \triangleright \ldots \triangleright 2\cdot 3 \triangleright 2\cdot 5 \triangleright 2\cdot 7 \triangleright \ldots \triangleright 2^2\cdot 3 \triangleright 2^2\cdot 5 \triangleright 2^2\cdot 7 \triangleright \ldots$
  $\triangleright\ 2^n\cdot 3 \triangleright 2^n\cdot 5 \triangleright 2^n\cdot 7 \triangleright \ldots \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1.$
  where $a \triangleright b$ means *a* precedes *b* in the order.
- Theorem: Suppose that F is a continuous function having a point x* of period m: $F^m(x^*) = x^*$. Then it has a point with period n if $m \triangleright n$.
- In particular, if x = F(x) has a 3-periodic solution (m = 3), then it has a periodic solution of every possible period.
- "Period three implies chaos," by Li and Yorke, 1975.
- With some windows in chaos where a periodic solution may be stable.



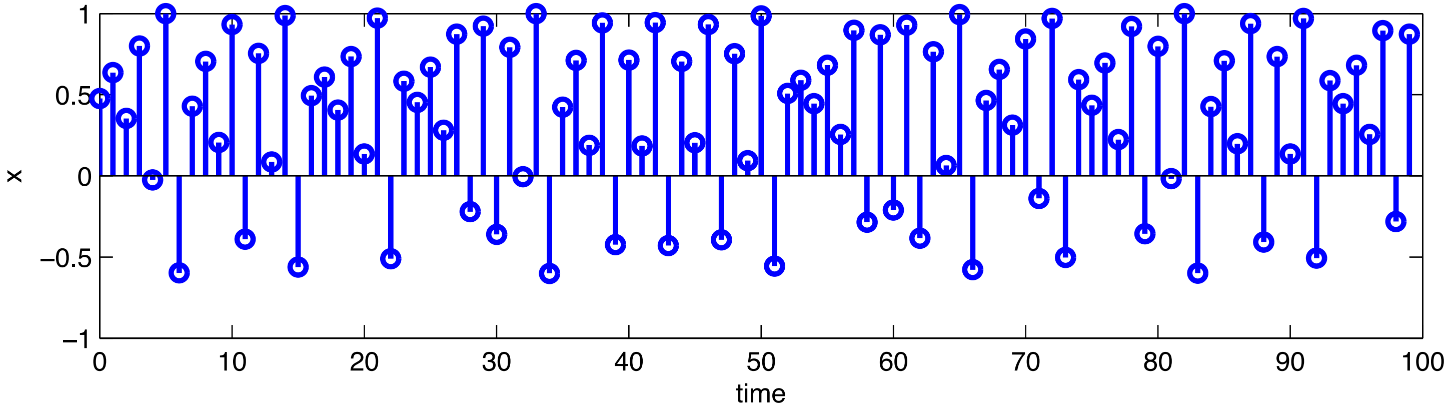
Period Doubling Route to Chaos

$$x(t+1) = 1 - \lambda x^2(t)$$



Property 3: Dense set of periodic solutions

$$x(t+1) = 1 - \lambda x^2(t)$$



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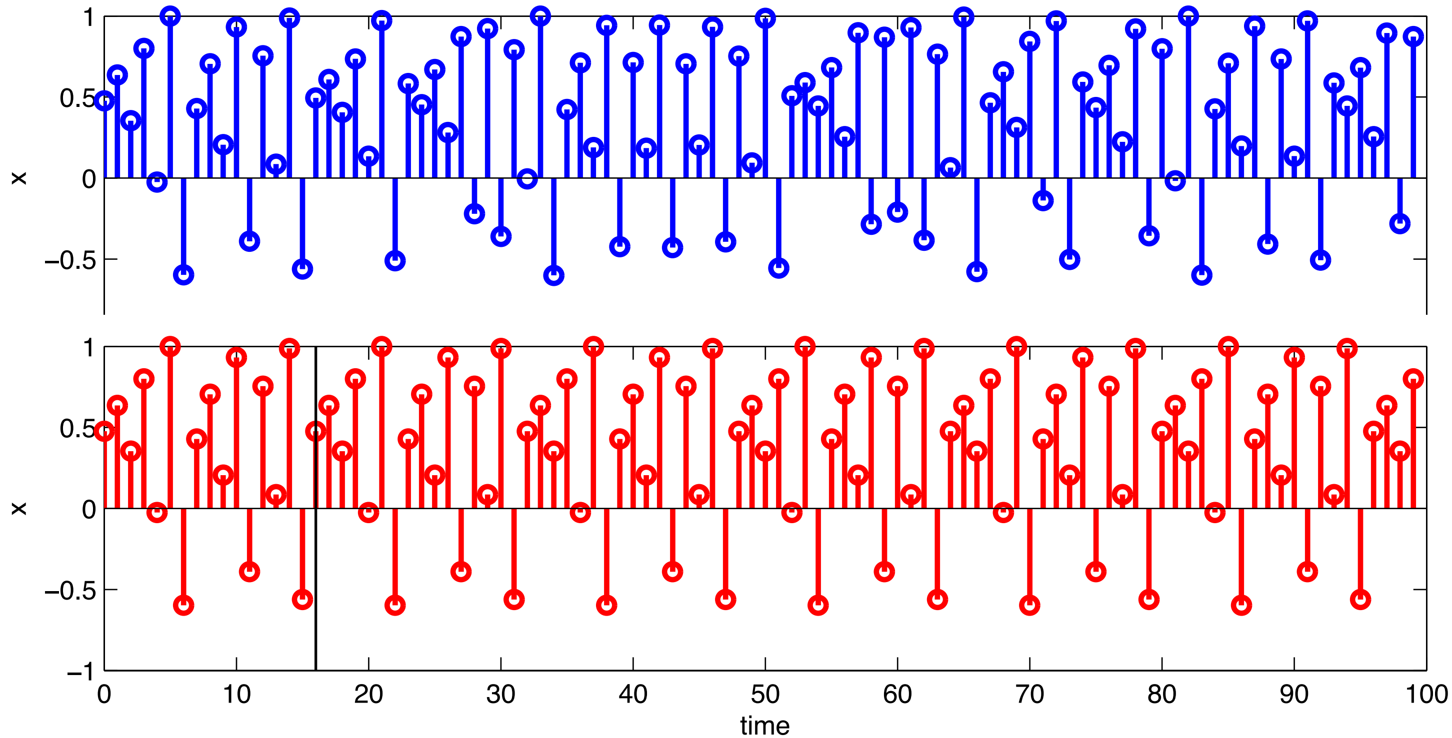

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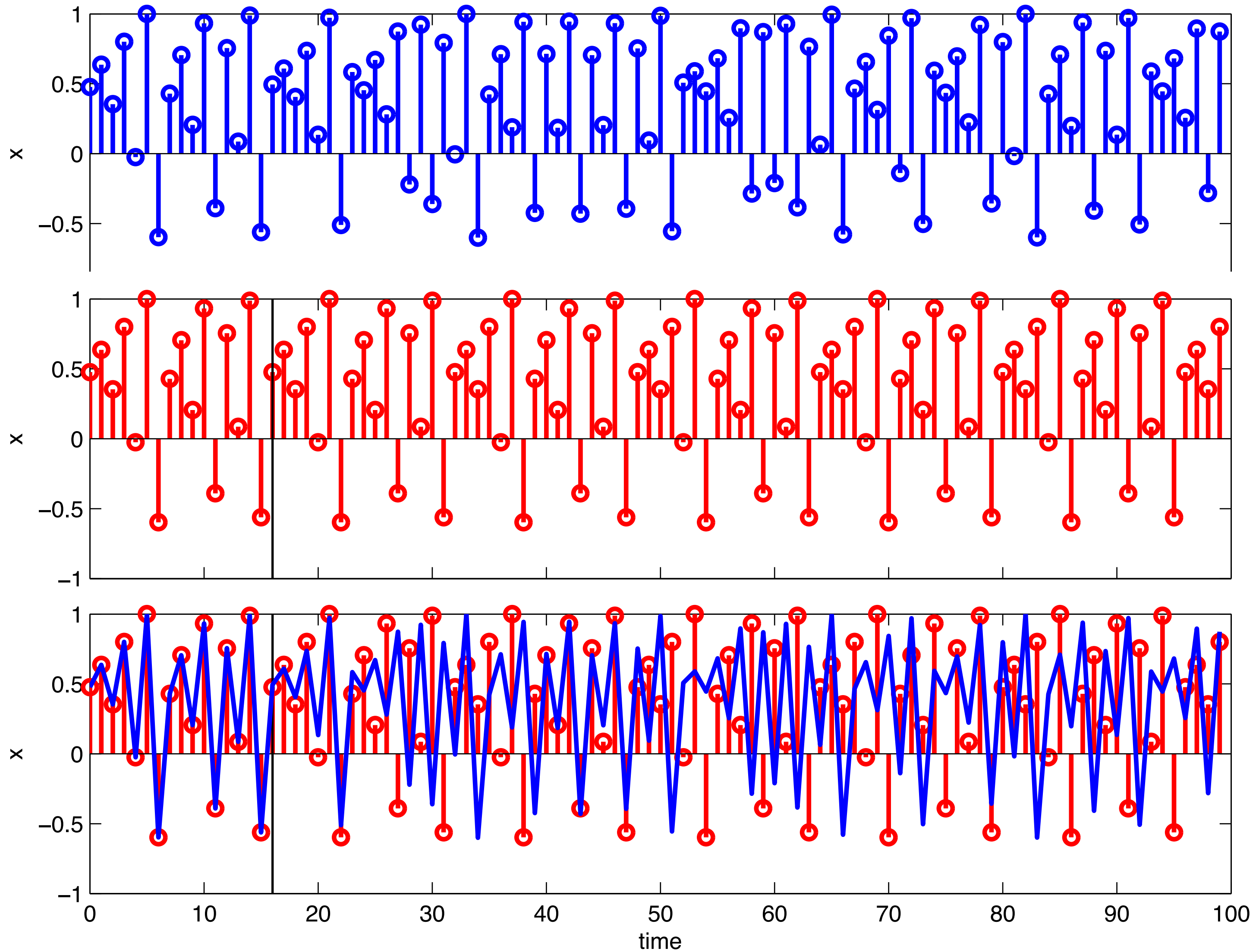

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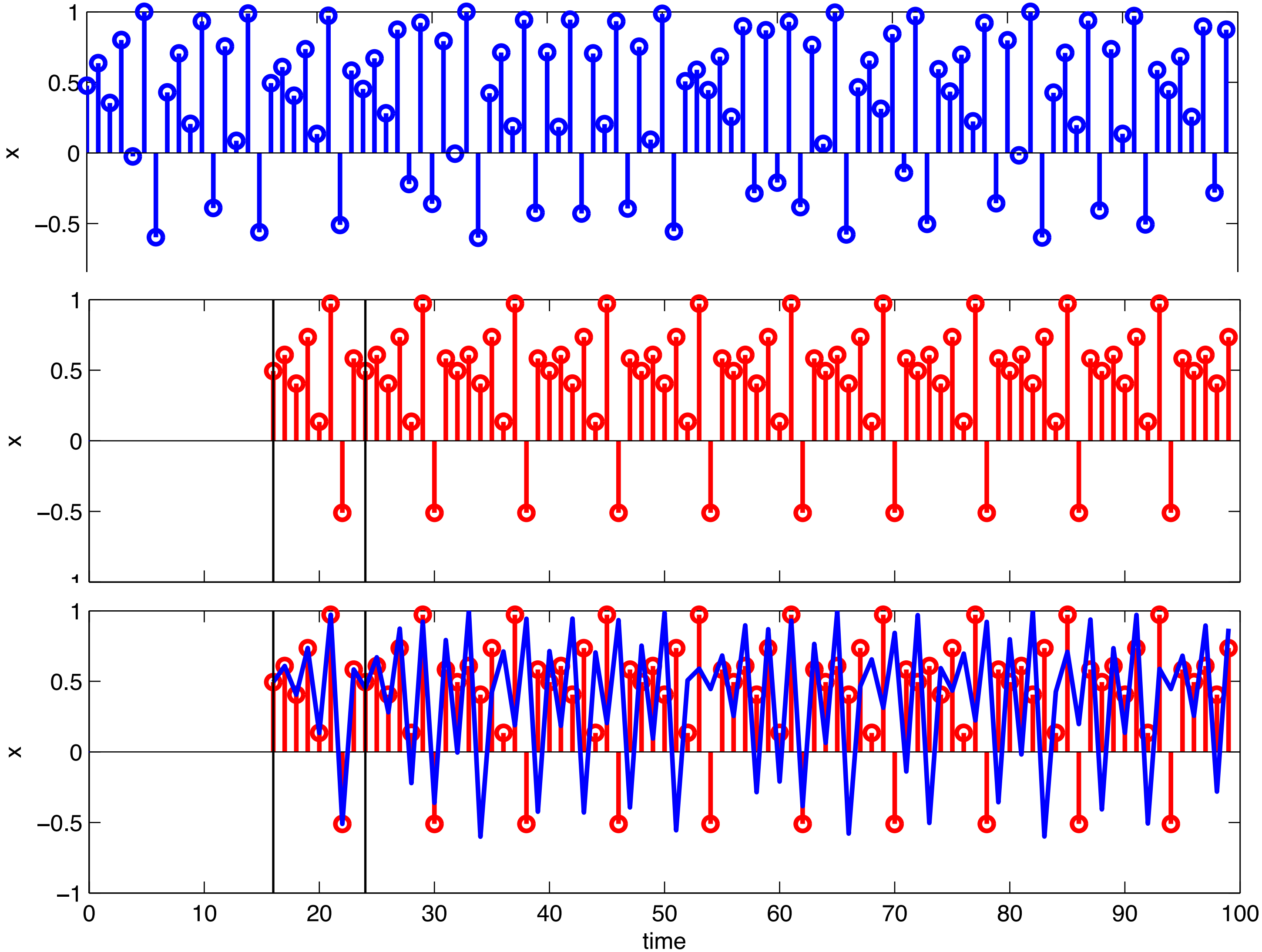

Property 3: Dense set of periodic solutions

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