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Physarum Can Compute Shortest Paths

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References

Material (articles and slides):

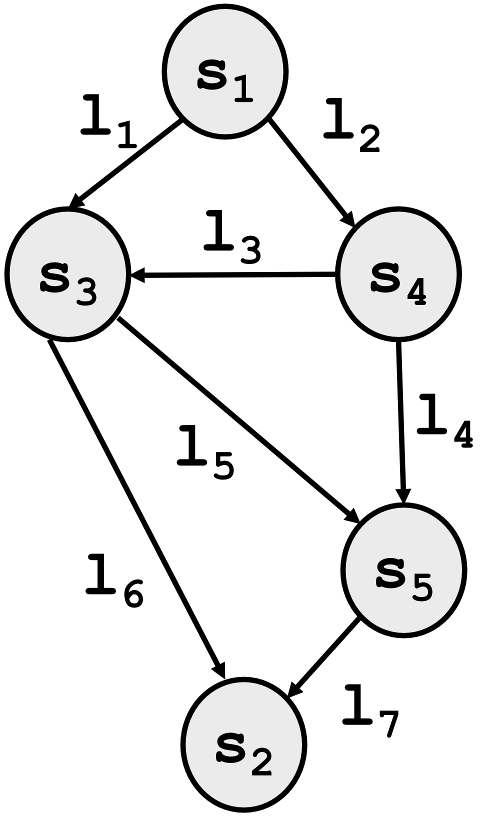
- ❑ Vincenzo Bonifaci, Kurt Mehlhorn and Girish Varma, ``Physarum can compute shortest paths", Journal of Theoretical Biology, vol. 309, pp. 121--133, 2012
- ❑ Vincenzo Bonifaci, ``Physarum can compute shortest paths: a shorter proof", Information Processing Letters, vol. 113, pp. 4-7, 2013.

Physarum Polycephalum

- Slime mold (myxomycète)
  - Unicellular
  - Multiple heads (nuclei)
- Eats spores, bacteria, etc

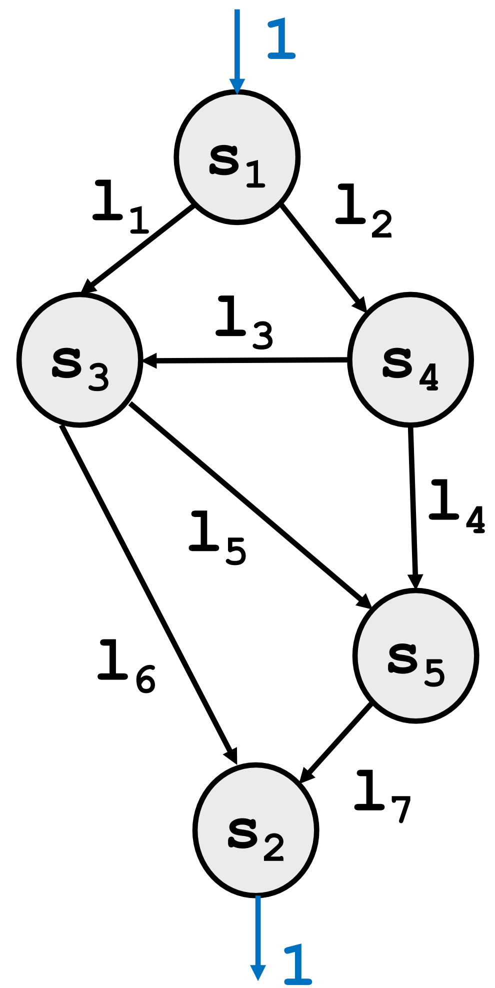
Model: Maze = Directed graph G(V,E)

- Directed Graph G(V,E):
  - m vertices $V = (s_1, s_2, \dots, s_m)$
  - n oriented edges $E = (1,2, \dots, n)$
  - Edge lengths $l = (l_1, l_2, \dots, l_m)$ with $l_i \neq l_j$ if $i \neq j$.
  - Edge variable “widths” = state of the system: $x = (x_1, x_2, \dots, x_n)$
  - Edge ”resistances” $r_i = l_i / x_i$
  - $R(x) = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & r_n \end{bmatrix} = \begin{bmatrix} l_1/x_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & l_n/x_n \end{bmatrix}.$



Model: Unit flow q routed in G(V,E)

- Unit flow q routed in the graph
  - $q = (q_1, q_2, \dots, q_n)$
  - Source = s_1 , sink = s_2. ($s_3, \dots, s_m$ = relays)
- Routing of q in G(V,E) imposes that $Aq = b$, where
  - A = adjacency matrix:
    - $A_{ij} = 1$ if node s_i is the origin of edge *j*
    - $A_{ij} = -1$ if node s_i is is the termination of edge *j*
    - $A_{ij} = 0$ if node s_i is not an endvertex of edge *j*
  - b = external input/output flows
    - $b_1 = 1$ because node s_1 is the source of the unit flow
    - $b_2 = -1$ because node s_2 is the sink of the unit flow
    - $b_i = 0$ for $3 \le i \le n$ because nodes $s_3, \dots, s_m$ are only relays (no fresh input or output).
- Flow q chosen to minimize $q^T R(x) q$ under constraint $Aq = b$.



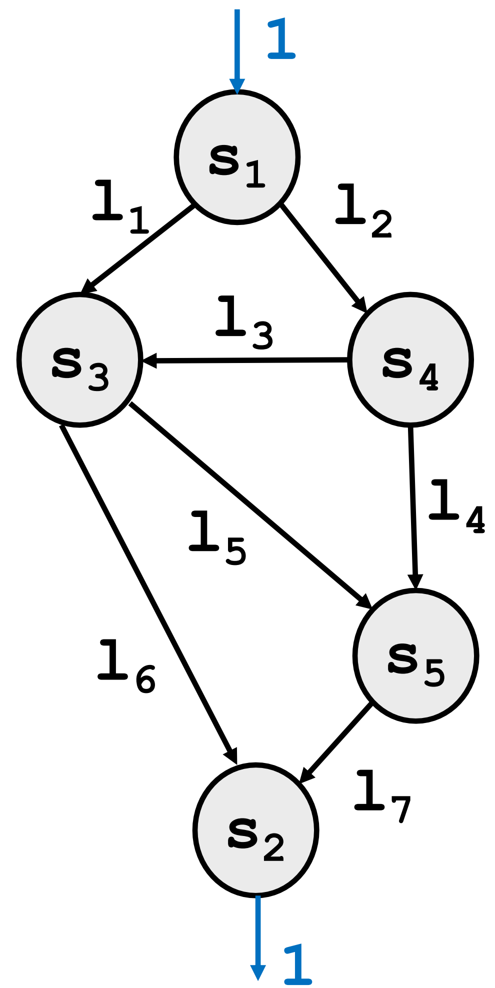
Model: Unit flow q routed in G(V,E)

- Flow $q(x)$ chosen to minimize $q^T R(x) q$ under constraint $Aq = b$.
- Dynamical system:

$$\dot{x} = |q(x)| - x$$

- Equilibrium: $x^* = |q(x^*)|$
  - $x_i = 1$: edge wide open (traversed by unit flow)
  - $x_i = 0$: edge closed (not traversed by unit flow).
- Initial state $x_i(0) > 0$.
- $\dot{x}_1 = |q_i(x)| - x_i \geq x_i$

$$\Longrightarrow x_i\ (t) \geq\ x_i(0)e^{-t} \geq 0 \text{ for } t \geq 0.$$



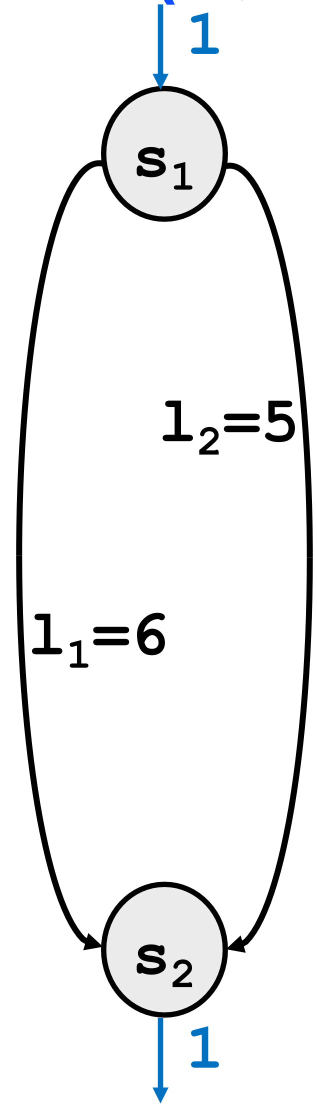
2-d Model: Unit flow q routed in G(V,E)

- Flow $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ chosen to minimize $q^T R(x) q$ under constraint $Aq = b$.
- $Aq = b \Longrightarrow q_1 + q_2 = 1.$
- $q^T R q = r_1 q_1^2 + r_1 q_2^2 = \frac{l_1 q_1^2}{x_1} + \frac{l_2 q_2^2}{x_2}$
- minimized by $q_1 = \frac{l_2 x_1}{l_1 x_2 + l_2 x_1}$

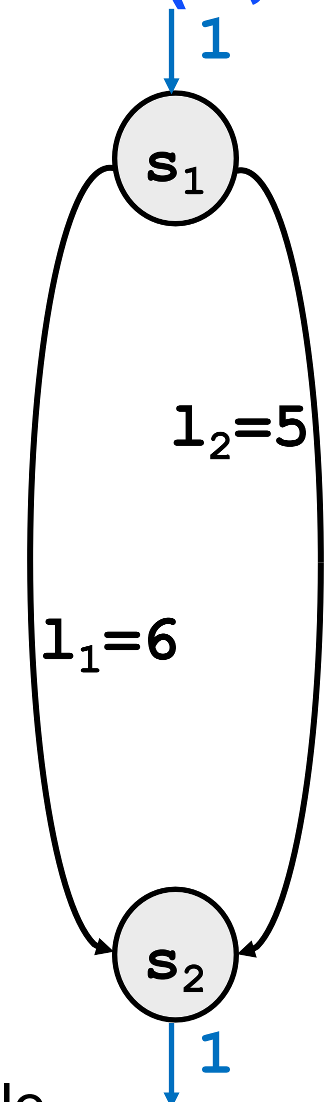
  $q_2 = \frac{l_1 x_2}{l_1 x_2 + l_2 x_1}$
- Dynamical system:

$$\dot{x}_1 = |q_1(x_1, x_2)| - x_1 = \frac{l_2 x_1}{l_1 x_2 + l_2 x_1} - x_1$$

$$\dot{x}_2 = |q_2(x_1, x_2)| - x_2 = \frac{l_1 x_2}{l_1 x_2 + l_2 x_1} - x_2$$



2-d Model: Unit flow q routed in G(V,E)



- Dynamical system:

$$\dot{x}_1 = \frac{l_2 x_1}{l_1 x_2 + l_2 x_1} - x_1$$

$$\dot{x}_2 = \frac{l_1 x_2}{l_1 x_2 + l_2 x_1} - x_2$$

- Equilibrium points are (1,0) and (0,1).
- Jacobian

$$J(1,0) = \begin{bmatrix} -1 & -{l_1}/{l_2} \\ 0 & -1 + {l_1}/{l_2} \end{bmatrix} \Rightarrow \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = -1 + {l_1}/{l_2} \end{matrix}$$

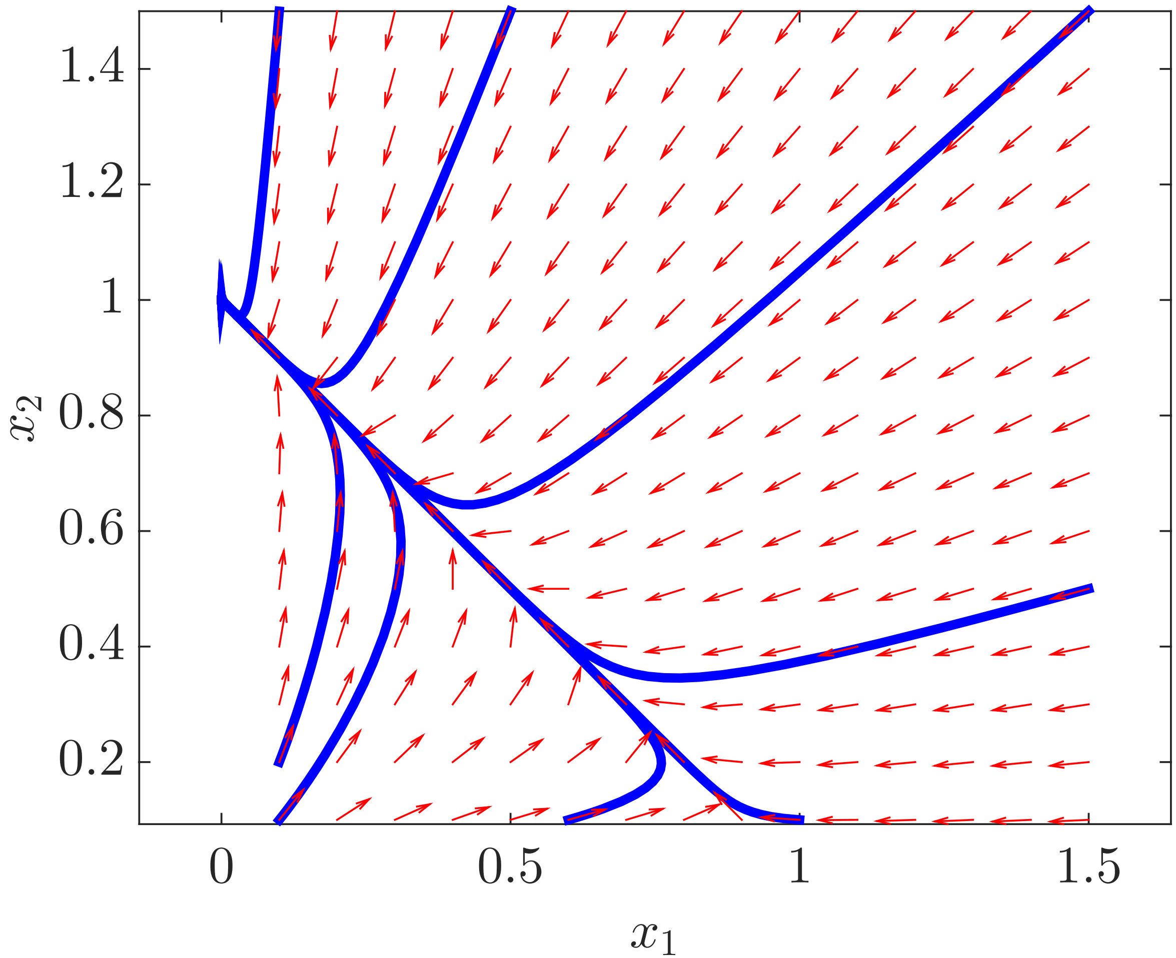
$$J(0,1) = \begin{bmatrix} -1 & 0 \\ -{l_2}/{l_1} & -1 + {l_2}/{l_1} \end{bmatrix} \Rightarrow \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = -1 + {l_2}/{l_1} \end{matrix}$$

- If $l_1 < l_2 < l_2$ then (1,0) = stable node; (0,1) = saddle.
- If $l_1 > l_2$ then (1,0) = saddle; (0,1) = stable node.

2-d model

- 2-d system such that $\dot{x}_1 + \dot{x}_2 = 1 - (x_1 + x_2)$
- Can restrict 2-d state space to

$$\Omega = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 1\}$$



Lyapunov function

- $W(x) = x^T R x - \min\{l_1, l_2\} = l_1 x_2 + l_2 x_1 - \min\{l_1, l_2\}$
  - W continuously differentiable,
  - $W(x_1, x_2) \geq \min\{l_1, l_2\}(x_1 + x_2 - 1) \geq 0$ for $(x_1, x_2) \in \Omega$,
  - $U_K = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0, l_1 x_2 + l_2 x_1 - \min\{l_1, l_2\} < K\}$ bounded,
  - $$\begin{aligned} \dot{W}(x) = l^T \dot{x} &= l^T |q| - l^T x \\ &= x^T R |q| - x^T R x \\ &= \left(x^T R^{1/2}\right) \cdot \left(R^{1/2} |q|\right) - x^T R x \\ &\leq (x^T R x)^{\frac{1}{2}} \cdot (q^T R q)^{\frac{1}{2}} - x^T R x \\ &\leq (x^T R x)^{\frac{1}{2}} \cdot (x^T R x)^{\frac{1}{2}} - x^T R x = 0 \end{aligned}$$
  - $\dot{W}(x) = 0 \Leftrightarrow R^{1/2} x$ and $R^{1/2}|q|$ are parallel

    $\Leftrightarrow x$ and $|q|$ are parallel

    $\Leftrightarrow x = |q|$ since $x_1 + x_2 = 1 = q_1 + q_2$, and all ≥ 0

    $\Leftrightarrow \dot{x} = |q| - x = 0$

    $\Leftrightarrow x^* = |q|$ is an equilibrium point = (1,0) or (0,1).