Dynamical Systems For Engineers

Test 1

School I&C, Master Course

NAME and First name:

Your answers are to be written in the space provided just after each question, hence if a page is unstapled, please mark your name on it. There is a total of 7 pages. Your answers and justifications must be clear, precise and complete. The notation $\dot{x}$ stands for dx/dt.

Maximum: 20 points

Question 1 (6 points)

Consider a continuous-time dynamical system, with state $x \in \mathbb{R}$, whose state equation is

$$\dot{x} = -\alpha x + |x+1| - |x-1|,$$

where $\alpha \in \mathbb{R}$ is a parameter.

1. (1pt) Does the system admit one unique solution $x(t)$ for each initial state $x(0) \in \mathbb{R}$ and for any $\alpha \in \mathbb{R}$? Justify your answer.

Solution: Let

$$\dot{x} = F(x) = -\alpha x + |x+1| - |x-1|. \tag{1}$$

The system admits one unique solution $x(t)$, if $F(x) = -\alpha x + |x+1| - |x-1|$ is continuous and locally Lipschitz. Clearly, $F(x)$ is continuous, since it is sum of three continuous functions.

To prove that it is Lipschitz, we show that each term of $\dot{x} = F(x)$ in (1) is Lipschitz, the sum of Lipschitz functions is easily shown to be Lipschitz by the triangle inequality.

1) For the first term of $F(x)$ (i.e., $-\alpha x$), since $|\alpha(x-x')| = |\alpha||(x-x')|$, we can choose $k_1 = |\alpha|$ (the slope is $|\alpha|$).

2) For the second term of $F(x)$ (i.e., $|x+1|$), the magnitude of the slope is always less than 1, so it is Lipschitz.

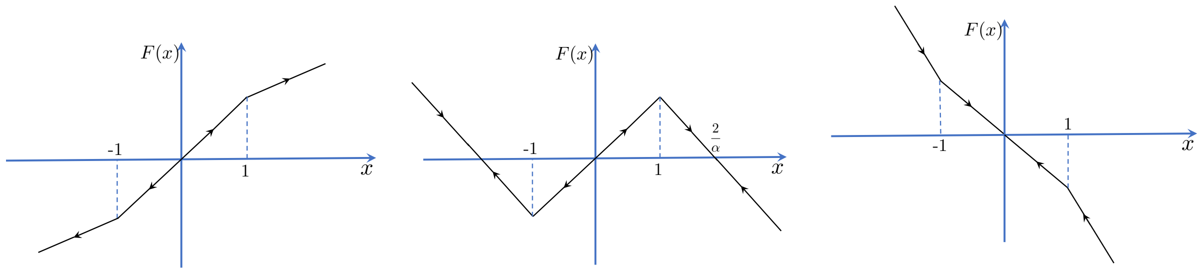
3) Similarly, for the third term of $F(x)$ (i.e., $-|x-1|$), the magnitude of the slope is always less than 1, so it is Lipschitz.

We conclude that $F(x)$ is Lipschitz and therefore that the system admits a unique solution $x(t)$ for each initial state $x(0) \in \mathbb{R}$ and for any $\alpha \in \mathbb{R}$.

2. (1.5pt) The origin is an equilibrium of the system for all $\alpha \in \mathbb{R}$. Characterize whether it is asymptotically stable, stable or unstable equilibrium point, as a function of α (i.e., specify the corresponding range of values α for which your answer is valid). Justify your answer.

Solution:

For $|x| < 1$, we have $\dot{x} = F(x) = (2-\alpha)x$. Hence, the Jacobian for $x = 0$ is $J(0) = 2 - \alpha$. The origin is an asymptotically stable equilibrium point if $\alpha > 2$ (since $J(0) < 0$), and it is an unstable equilibrium point if $\alpha < 2$ (since $J(0) > 0$). For $\alpha = 2$ the origin is not an hyperbolic equilibrium point, but since $\dot{x} = F(x) = 0$ for all $x \in [-1, 1]$ and thus for a neighborhood around 0, the origin is a stable equilibrium point.



(a) $\alpha \leq 0$ (b) $0 < \alpha \leq 2$ (c) $\alpha > 2$

Figure 1: $\dot{x} = F(x)$ for different values of α.

3. (2pts) Let $\xi(t)$ denote the solution of this system with initial condition $\xi(0) = 3$. What is its ω-limit set $\mathcal{S}_\omega(\xi)$, as a function of α ? Justify your answer.

Solution:

First, we specify $\dot{x} = F(x)$ for different values of x.

$$\dot{x} = F(x) = \begin{cases} -\alpha x + 2, & \text{if } x \geq 1, \\ (2-\alpha)x, & \text{if } |x| < 1, \\ -\alpha x - 2, & \text{if } x \leq -1. \end{cases}$$

The initial condition $\xi(0) = 3$ falls in the first range $x \geq 1$. In this range, for $\alpha \leq 0$, $\dot{x} = F(x) > 0$ and $\xi(t) \to \infty$, hence for $\alpha < 0$, the ω-limit set is $\mathcal{S}_\omega(\xi) = \emptyset$ (see Figure 1a).

For $0 < \alpha \leq 2$, the ω-limit set is $\mathcal{S}_\omega(\xi) = \{2/\alpha\}$. The reason is that for $x > 2/\alpha$ we have $\dot{x} = F(x) < 0$, for $1 < x < 2/\alpha$ we have $\dot{x} = F(x) > 0$, and for $x = 2/\alpha$ we have $\dot{x} = F(x) = 0$ (see Figure 1b).

For $\alpha > 2$, the ω-limit set is $\mathcal{S}_\omega(\xi) = \{0\}$, because $\xi(t) \to 0$ for any $\xi(0) \in \mathbb{R}$ (see Figure 1c).

4. (1.5pts) List all the attractors of this system, if any, as a function of α. Justify your answer.

Solution:

Recall that

$$\dot{x} = \begin{cases} -\alpha x + 2, & \text{if } x \geq 1, \\ (2-\alpha)x, & \text{if } |x| < 1, \\ -\alpha x - 2. & \text{if } x \leq -1. \end{cases}$$

According to the definition of attractor, for $0 < \alpha < 2$ the attractors are $\{2/\alpha, -2/\alpha\}$, because there exists an open set $\mathcal{U}$ around them such that all solutions starting in $\mathcal{U}$ converge to $\{2/\alpha, -2/\alpha\}$ as $t \to \infty$ (see Figure 1b). For example, for the attractor $2/\alpha$, one consider $\mathcal{U} = (1/\alpha, 4/\alpha)$.

For $\alpha > 2$, as it is seen in 3. any $x(0)$ will eventually converge to the origin 0 as $t \to \infty$, hence the origin is an attractor for $\alpha > 2$ (see Figure 1c).

For $\alpha = 2$, the closed interval $[-1, 1]$ is the only attractor, because it is the smallest compact set that is forward invariant (since $\dot{x} = 0$ for all $x \in [-1, 1]$) and surrounded by an open set $\mathcal{U} = \mathbb{R} \setminus [-1, 1]$ such that any point starting in $\mathcal{U}$ will converge to -1 or $+1$ (depending on the value of $x(0)$).

Question 2 (3 points)

Consider an autonomous discrete-time linear system in $\mathbb{R}^2$ given by

$$\begin{aligned} x_1(t+1) &= \alpha x_1(t) - x_2(t) \\ x_2(t+1) &= x_1(t) \end{aligned}$$

where $\alpha \in \mathbb{R}$ is a parameter. Characterize its stability (i.e. asymptotic stable, stable, weakly unstable, strongly unstable), as a function of α (i.e., specify the corresponding range of values α for which your answer is valid). Justify your answer.

Solution:

The matrix that describes the system is

$$A = \begin{bmatrix} \alpha & -1 \\ 1 & 0 \end{bmatrix}.$$

The system is discrete, so we must compare

The characteristic polynomial is $\chi(\lambda) = \lambda(\lambda-\alpha)+1 = \lambda^2-\alpha\lambda+1$. Its discriminant is $\Delta = \alpha^2-4$.

If $|\alpha| > 2$, then $\Delta > 0$ and A has two distinct eigenvalues $\lambda_\pm = (\alpha \pm \sqrt{\Delta})/2$. If $\alpha > 2$, then $\lambda_+ > 1$. If $\alpha < -2$, then $\lambda_- < -1$. In both cases, this means that the system is strongly unstable.

If $|\alpha| < 2$, then $\Delta < 0$ and A has two distinct eigenvalues $\lambda_\pm = (\alpha \pm j\sqrt{-\Delta})/2$. The module of the eigenvalues is $|\lambda_\pm| = \sqrt{(\alpha^2 + 4 - \alpha^2)/4} = 1$. This means that the system is stable, but not asymptotically stable.

If $|\alpha| = 2$, then $\Delta = 0$ and A has a single eigenvalue $\lambda = \alpha/2$. A is not a multiple of the identity matrix, which means that it is not diagonalizable and therefore it has a Jordan block of dimension 2. Consequently, the system is weakly unstable.

Question 3 (5 points)

The state equations of an autonomous continuous-time nonlinear system in $\mathbb{R}^2$ are

$$\begin{aligned}\dot{x}_1 &= -x_2 x_1^2\\ \dot{x}_2 &= x_1^3.\end{aligned}$$

1. (1pt) Does this system have bounded solutions ? Justify your answer.

Solution:

Let the Lyapunov function be $W(x) = x_1^2 + x_2^2$. Clearly, $W(x)$ is non-negative for all $x \in \mathbb{R}^2$ and the level sets of $W(x)$ are bounded. As $\dot{W}(x) = 0$ for all $x \in \mathbb{R}^2$ (W is non-increasing along trajectories), the system has bounded solutions.

2. (1pt) Compute all equilibrium points of the system.

Solution:

By setting $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, we see that any point $x = (0, x_2)$, where $x_2 \in \mathbb{R}$, is an equilibrium point of the system.

3. (1pt) Let $\xi(t) = (\xi_1(t), \xi_2(t))$ denote the solution of this system with initial condition $\xi(0) = (\xi_1(0), \xi_2(0)) = (3, 4)$. What is its ω-limit set $\mathcal{S}_\omega(\xi)$? Justify your answer.

Solution:

Using polar coordinate transformation we have, $r^2 = x_1^2 + x_2^2$. From 1. we know that $\dot{r} = 0$. For $x_2 > 0$ and $x_1 \neq 0$ we have $\dot{x}_1 < 0$, and for $x_1 = 0$ we have $\dot{x}_1 = 0$. This means that $\xi(t)$ turns counter-clockwise and converges to the point $x = (0, 5)$ (as $r = \sqrt{3^2 + 4^2}$ needs to remain fixed).

4. (1pt) Does this system have uniformly asymptotically bounded solutions? Justify your answer.

Solution:

According to the definition of uniformly asymptotically bounded systems, there should be a $B > 0$ such that for each $x(0) \in \mathbb{R}^2$, there is a finite time $T > 0$ such that for all $t \geq T$

$$\|x(t)\| \leq B.$$

However, here since $\dot{r} = 0$, there is no B for which we can guarantee that for every solution $x(t)$, $\|x(t)\| \leq B$. In fact, for any $B > 0$ if we choose $\|x(0)\| > B$ then for all t we have $\|x(t)\| > B$. Thus, the system does not have uniformly uniformly asymptotically bounded solutions.

5. (1pt) Characterize the stability (i.e., asymptotically stable, stable or unstable) of the equilibrium points of the system. Justify your answer.

Solution:

Since the equilibrium points $x^\star = (0, x_2)$ are not hyperbolic, we cannot use the linearizion technique. However, following the same reasoning as in parts 3. and 4., we know that $r(t) = r(0)$ for all t.

Next, let

$$\varphi = \arctan\left(\frac{x_2}{x_1}\right).$$

We find

$$\dot{\varphi} = \frac{x_1\dot{x_2} - x_2\dot{x_1}}{x_1^2 + x_2^2},$$

and after easy computations, $\dot{\varphi} = x_1^2$. This means that φ consistently increases (i.e., $x(t) = (x_1(t), x_2(t))$ turns counter-clockwise) and it converges to the equilibrium point that lies on the line $x_1 = 0$ (and thus $\dot{\varphi} \to 0$). In other words, if $x_1(0) > 0$ at $t = 0$, then $x(t) \to (0, r(0))$, and if $x_1(0) < 0$ at $t = 0$, then $x(t) \to (0, -r(0))$. As a result, the origin is a stable equilibrium point, since the distance between the origin and $x(t)$ remains fixed $r(t) = r(0)$. The other equilibrium points $x^\star = (0, x_2)$ for $x_2 \in \mathbb{R} \setminus \{0\}$ are unstable: Let $x_2 > 0$ and $\epsilon > 0$; a point $x(t)$ with initial condition $x(0) = (-\epsilon, x_2)$ will converge to $\left(0, -\sqrt{x_2^2 + \epsilon^2}\right)$, i.e., far from $(0, x_2)$. If $\epsilon > 0$ is such that $\epsilon < 2r(0)$ we cannot find a $\delta > 0$ such that for any solution x with $\|x(0) - x^\star\| \leq \delta$, we have for all $t \geq 0$ $\|x(t) - x^\star\| \leq \epsilon$. A similar argument holds for $x_2 < 0$, $\epsilon > 0$, and a point $x(t)$ with initial condition $x(0) = (+\epsilon, x_2)$.

Question 4 (6 points)

Consider an autonomous continuous-time nonlinear system in $\mathbb{R}^2$ given by

$$\begin{aligned}\dot{x}_1 &= x_1(1-x_1)(1+x_2)\\ \dot{x}_2 &= x_1^3 - x_2.\end{aligned}$$

1. (1.5pts) Find all equilibrium points of this system, and characterize their stability (i.e. asymptotically stable, stable, unstable).

Solution:

We have

$$\dot{x_1} = 0 \iff x_1(1-x_1)(1+x_2) = 0 \iff x_1 = 0 \text{ or } x_1 = 1 \text{ or } x_2 = -1.$$

With the condition $\dot{x_2} = 0$, we find three equilibrium point, $(0,0)$, $(1,1)$ and $(-1,-1)$.

The Jacobian can be computed as

$$J(x) = \begin{bmatrix}(1+x_2)(1-2x_1) & x_1(1-x_1)\\ 3x_1^2 & -1\end{bmatrix}.$$

The Jacobian in $(0,0)$

$$J((0,0)) = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

has a positive eigenvalue 1, thus $(0,0)$ is an unstable equilibrium point.

The Jacobian in $(1,1)$

$$J((1,1)) = \begin{bmatrix}-2 & 0\\ 3 & -1\end{bmatrix}$$

has two strictly negative eigenvalues -2 and -1, thus $(1,1)$ is an asymptotically stable equilibrium point.

The Jacobian in $(-1,-1)$

$$J((-1,-1)) = \begin{bmatrix}0 & -2\\ 3 & -1\end{bmatrix}$$

has two eigenvalues $\lambda_\pm = (-1 \pm j\sqrt{23})/2$ that verify $\Re(\lambda_\pm) < 0$, thus $(-1,-1)$ is an asymptotically stable equilibrium point.

2. (1pt) Let $S_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0\} = \{(x_1, 0) \mid x_1 \in \mathbb{R}\}$. Is S_1 forward invariant? Is S_1 backward invariant? Justify your answer.

Solution:

Let $a > 0$; if a solution $x(t) = (x_1(t), x_2(t))$ is such that $x(t_0) = (a, 0) \in S_1$ for some $t_0 \in \mathbb{R}$, then we have

$$\begin{aligned}\dot{x_1}(t_0) &= a(1-a)\\ \dot{x_2}(t_0) &= a^3,\end{aligned}$$

which means that $\dot{x_2}(t_0) > 0$, i.e., $x_2(t_0 + \epsilon) > 0$ for $\epsilon > 0$ small enough, which means that $x(t_0 + \epsilon) \notin S_1$: S_1 is not forward invariant.

Similarly, $x(t_0 - \epsilon) \notin S_1$: S_1 is not backward invariant.

3. (1pt) Let $S_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\} = \{(0, x_2) \mid x_2 \in \mathbb{R}\}$. Is S_2 forward invariant? Is S_2 backward invariant? Justify your answer.

Solution: Let $x_0 \in S_2$, i.e., $x_0 = (0, a)$ for some $a \in \mathbb{R}$, and let $x(t) = (x_1(t), x_2(t))$ be such that $x(t_0) = x_0 = (0, a)$ for some $t_0 \in \mathbb{R}$. We have $\dot{x_1}(t_0) = 0$; i.e., x_1 stays constant. This means that necessarily, for every $t \in \mathbb{R}$, $x_1(t) = x_1(t_0) = 0$, i.e., $x(t) \in S_2$: S_2 is invariant (it is both forward and backward invariant).

4. (2.5 pts) Sketch the phase portrait of the system as accurately as possible.