Dynamical Systems for Engineers: Exercise Set 10

Exercise 1

Consider the autonomous nonlinear system

$$\begin{aligned}\dot{x_1} &= -x_2 + \lambda x_1(x_1^2 + x_2^2 - 1)\\ \dot{x_2} &= x_1 + \lambda x_2(x_1^2 + x_2^2 - 1).\end{aligned}$$

1. By applying the Poincaré-Bendixson theorem, check that if $\lambda < 0$, this system has a periodic solution in $\mathcal{M} = \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 - \varepsilon < r^2 < 1 + \varepsilon\}$. Verify that this periodic solution is $\xi(t) = (\cos t, \sin t)$ (up to a time shift $0 \leq \tau < 2\pi$, i.e any function $x_\tau(t) = \xi(t + \tau)$ is also a periodic solution), and that it exists for any $\lambda \in \mathbb{R}$.

2. Write the variational equations of this system

$$\dot{M}(t) = J\left(\xi(t)\right) M(t) \tag{1}$$

in the coordinates (x_1, x_2).

3. In the polar coordinates $(r, \theta) \in \mathbb{R}^+ \times [0, 2\pi)$, defined by

$$r = \left(x_1^2 + x_2^2\right)^{1/2} \tag{2}$$

$$\varphi = \arctan\left(\frac{x_2}{x_1}\right), \tag{3}$$

the system is easier to analyze. Let $\xi_{\text{polar}}(t)$ denote the periodic solution $\xi(t)$ expressed in the polar coordinates. Write the variational equation of the system

$$\dot{M}_{\text{polar}}(t) = J\left(\xi_{\text{polar}}(t)\right) M_{\text{polar}}(t) \tag{4}$$

in the polar coordinates (**??**), (**??**).

4. Compute the Floquet multipliers of this periodic solution, and deduce the parameter range λ for which it is (un)stable.

Exercise 2

We consider the following planar system:

$$\begin{aligned}\dot{x_1} &= x_1 - x_2 - x_1^3\\ \dot{x_2} &= x_1 + x_2 - x_2^3.\end{aligned}$$

1. Does this system have a periodic solution that lies within the region

$$\mathcal{M}_1 = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid \|x\| < 2/3\right\} ?$$

2. Find an annular region

$$\mathcal{M}_2 = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid r_2 < \|x\| < r_1\right\}$$

where the system has a periodic solution. Hint: think of the polar change of variables (**??**), (**??**).