Dynamical Systems
for Engineers: Exercise Set 7

Exercise 1

In the context of immunology, we would like to analyze an infection model. Consider two types of cells: the target cells T that might become infected, and the cells I infected by some virus.

The following events affect the cell populations sizes:

- target cells are produced at a constant rate σ cells per time unit;
- a fixed fraction δ_T of all target cells dies every unit of time;
- the infected cells have the ability to infect target cells when they encounter them; assume that target cells become infected at rate β per time unit per infected cell, in other words, a proportion β of all infected cells will infect the target cells in a time unit;
- a fixed fraction δ_I of all infected cells dies every unit of time.

Of course, all parameters are positive.

1. Write the differential equations describing the dynamics of the populations of target (T) and infected (I) cells. Note that the state space is $\mathbb{R}^2_+$, because $T(t), I(t) \geq 0$ for all $t \in \mathbb{R}_+$ by definition.

Hint: the system has the form

$$\begin{aligned} \frac{dT}{dt}(t) &= \sigma - \delta_T \cdot T(t) - f(\beta, T(t), I(t)) \\ \frac{dI}{dt}(t) &= f(\beta, T(t), I(t)) - \delta_I \cdot I(t) \end{aligned}$$

where f is a function that you should find from the modelling assumptions.

2. Find the equilibrium points of this system and give a biological meaning of each of them. In which steady state do you prefer to be?

3. Analyze the stability of the equilibrium points of this system.

Exercise 2

Consider the continuous-time autonomous nonlinear system

$$\begin{aligned} \dot{x}_1(t) &= \alpha x_1(t) + x_2(t)(1 + x_1(t)) \\ \dot{x}_2(t) &= -x_1(t)(1 + x_1(t)), \end{aligned}$$

where $\alpha \in \mathbb{R}$ is a parameter. Here, we assume that $\alpha \neq 0$.

1. Find the range of values of $\alpha \in \mathbb{R} \setminus \{0\}$, if any, for which the origin is a (locally) asymptotically stable equilibrium point of the system. Justify thoroughly your answer.

2. If $\alpha \neq 0$, find the range of values of $\alpha \in \mathbb{R} \setminus \{0\}$, if any, for which the origin is a *globally* asymptotically stable equilibrium point of the system.

Exercise 3

We consider a friction-less pendulum, as in Figure 1. A massless stiff rod of length l is fixed at an axis at one end and it has a mass m on the other end. Suppose the rotational motion is frictionless, and that the only force acting on the pendulum is gravity. Denoting by φ the angle between the rod and the vertical axis, Newton's law implies that the force that is applied to the pendulum is therefore $F = -mg \sin \varphi$.

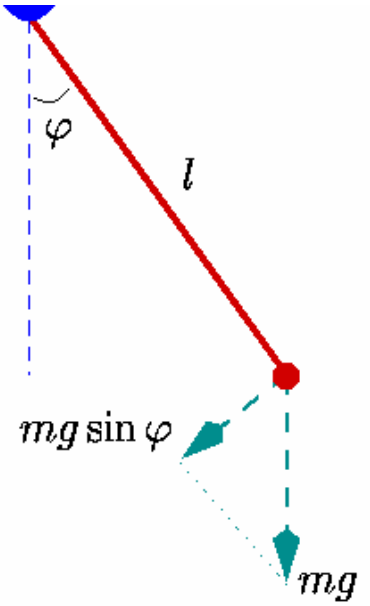

Figure 1: Pendulum system

1. Give the state equations of the friction-less pendulum system.

2. Compute the equilibrium point(s) of the system.

3. Characterize the stability of the equilibrium point(s).

4. Show that this is a Hamiltonian system.