Dynamical Systems
for Engineers: Exercise Set 5

Exercise 1

Consider the autonomous nonlinear system

$$\begin{aligned} \dot{x_1} &= -x_2 + \alpha x_1(x_1^2 + x_2^2 - 1) \\ \dot{x_2} &= x_1 + \alpha x_2(x_1^2 + x_2^2 - 1). \end{aligned}$$

1. For which value(s) of α can you guarantee that this system has

- asymptotically uniformly bounded solutions?
- bounded solutions, but that are not asymptotically uniformly bounded solutions?

Find a Lyapunov function that support your claim when your answer to the above questions is positive.

2. Verify your answer to question 1 by solving this system using the polar coordinates transformation that we have used before in class.

Exercise 2

The state equations of an autonomous continuous-time linear system in $\mathbb{R}^2$ are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \ln\left(x_1 + \sqrt{x_1^2 + 1}\right). \end{aligned}$$

Note: The function $\ln(x + \sqrt{x^2 + 1})$ is also known as $\text{Arcsh}(x)$. It is an odd function of $x \in \mathbb{R}$ and its primitive (indefinite integral) is

$$\int \ln\left(x + \sqrt{x^2 + 1}\right) dx = x \ln\left(x + \sqrt{x^2 + 1}\right) - \sqrt{x^2 + 1}.$$

Does this system have bounded solutions?

Exercise 3

Consider the autonomous nonlinear system (a variant of the Lozi map),

$$\begin{aligned} x_1(t+1) &= \alpha - 1 - \alpha\,|x_1(t)| + x_2(t) \\ x_2(t+1) &= \beta x_1(t) \end{aligned}$$

with $\alpha = 1.7$ and $\beta = 0.5$. Does this system have bounded solutions? Prove your claim. (Hint: try a few simulations, or check what happens when you compute the first iterations with an initial condition such that $x_1(0) < -K, x_2(0) < 0$, for a constant $K > 2$).