Dynamical Systems for Engineers: Exercise Set 2

Exercise 1

In the special case of a 1-dimensional ODE, the qualitative behavior of a solution can be determined by considering for each value of the state x the sign of the derivative of x. Consider a continuous-time dynamical system, with state $x \in \mathbb{R}$, whose state equation is the nonlinear ordinary differential equation

$$\dot{x}(t) = -x^3(t) + x(t).$$

1. Discuss the qualitative behavior of the solutions of this dynamical system. What are their $\omega-$ and α-limit sets? (Hint: try to find the roots of the right hand side of that equation).

2. What are the invariant sets of this system? Give an example of a set that is forward invariant but not invariant.

3. What is (are) the attractor(s) of this system, if any?

Exercise 2

Let $\xi(t)$ be the solution of the kth order ordinary differential equation

$$\frac{d^k\xi}{dt^k}(t) + a_1\frac{d^{k-1}\xi}{dt^{k-1}}(t) + \ldots + a_{k-1}\frac{d\xi}{dt}(t) + a_k\xi(t) = u(t),$$

where $u(t)$ is an input signal, and where $a_i \in \mathbb{R}$ for all $1 \leq i \leq k$. Show that this o.d.e. can be recast under the canonical state equation representation

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t)$$

where x is the state vector. Explicit x, A and B as functions of ξ, $d^i\xi/dt^i$ and a_i for $1 \leq i \leq k$. What is the dimension of this system?

Exercise 3

Consider the autonomous linear system

$$\begin{aligned}\dot{x_1}(t) &= x_1(t) + x_2(t) \\ \dot{x_2}(t) &= 4x_1(t) - 2x_2(t).\end{aligned}$$

1. Characterize the stability of this system.

2. Find the solution of the system subject to the initial conditions $(x_1(0), x_2(0)) = (2, -3)$.

3. Find the solution of the system subject to the initial conditions $(x_1(0), x_2(0)) = (1, -4)$.

Exercise 4: Fibonacci dynamical system

The Fibonacci numbers are given by the recurrence relation

$$\begin{aligned} F_0 &= 0,\ F_1 = 1 \\ F_{n+2} &= F_{n+1} + F_n,\ n > 0. \end{aligned}$$

As $n \to \infty$, the ratio of successive terms of the Fibonacci sequence tends to the golden ratio $\phi \doteq (1+\sqrt{5})/2$. In this exercise, we study the Fibonacci system and prove that any Fibonacci number can be given from a closed-form expression which is a function of ϕ. In the following, to facilitate the computations, express all the quantities that involve $\sqrt{5}$ as a function of ϕ.

1. Show that the above recurrence relation can be recast under the canonical state equation representation

$$x(t+1) = Ax(t), \tag{1}$$

where x is the state vector with dimension 2. Explicit x and A.

2. Prove that, as expected, system (1) is strongly unstable.

3. Find the solution of system (1) subject to the initial conditions given by the first two Fibonacci numbers, i.e., $(x_1(0), x_2(0)) = (0, 1)$.

4. Given the solution at previous step, show that the n_{th} Fibonacci number is given by

$$F_n = \phi \frac{\phi^n - (1-\phi)^n}{1+\phi^2},\ n \geq 0.$$

5. Show that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi.$$