Dynamical Systems for Engineers: Exercise Set 12

Exercise 1

We continue the study of the continued fraction map, given by

$$F(x) = \frac{1}{x} \mod 1 = \begin{cases} \frac{1}{x} - n & \text{if} \quad \frac{1}{n+1} < x \leq \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{if} \quad x = 0. \end{cases} \tag{1}$$

by focusing on the discrete-time dynamical system

$$x(t+1) = F(x(t)) = \frac{1}{x(t)} \mod 1 = \begin{cases} \frac{1}{x(t)} - n & \text{if} \quad \frac{1}{n+1} < x(t) \leq \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{if} \quad x(t) = 0. \end{cases} \tag{2}$$

The name "continued fraction map" comes from the correspondence between the trajectories of this dynamical system and the corresponding sequences of integers that form the continued fraction expansion of some real x_0. Indeed, the continued fraction expansion of a real x_0 is the sequence of integers $[n_0, n_1, n_2, \ldots]$ that expands x_0 as

$$x_0 = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}}$$

This continued fraction expansion is obtained by taking $n_0 = \lfloor x_0 \rfloor \in \mathbb{Z}$ as the integer part of x_0 (which is negative if $x_0 < 0$), and the following values n_t for all $t \in \mathbb{N}$, as follows. $x(0)$ is the fractional part of x_0, which is $x(0) = x_0 - n_0 = x_0 \mod 1$. Then for $t \in \mathbb{N}$, $n_{t+1} \in \mathbb{N}$ is the integer part of $1/x(t)$, i.e.,

$$n_{t+1} = \lfloor 1/x(t) \rfloor,$$

and $x(t+1) \in [0,1)$ is fractional part of $1/x(t)$, i.e.,

$$x(t+1) = 1/x(t) - n_{t+1} = F(x(t)).$$

For example, if $x_0 = -11/5$ then $n_0 = -3$, $x(0) = 4/5$, $n_1 = 1$, $x(1) = 1/4$, $n_2 = 4$ and $x(2) = 0$ and indeed

$$x_0 = -\frac{11}{5} = -3 + \frac{1}{1 + \frac{1}{4}}.$$

In this example, the trajectory converges to the fixed point 0 of the map in two iterations. One can show that x_0 is a rational number if and only if its continued fraction expansion is finite. Therefore of x_0 is a rational number, the trajectory $x(t)$ will converge to 0 in a finite number of iterations.

1. The real $x_0 = (\sqrt{5}+1)/2 \approx 1.618$ is known as the *golden number*. What is the solution $x(t)$ of (2) for $x(0) = x_0 \mod 1$, and what is the corresponding continued fraction expansion $[n_0, n_1, n_2, \ldots]$? What is the Lyapunov exponent of that particular solution $x(t)$?

2. We saw in that the map (1) is measure-preserving, with invariant measure being the Gauss measure whose density ($dP(x) = \rho(x)dx$) is

$$\rho(x) = \frac{1}{\ln 2} \frac{1}{1+x}.$$

It can be shown to be ergodic as well. What can you say about the Lyapunov exponent of almost all the solutions of the system (2) and their chaotic nature ? The following integral is useful:

$$\int_0^1 \frac{\ln x}{1+x} dx = -\frac{\pi^2}{12}.$$

Exercise 2

We saw two variants of the logistic map, which are

$$F : [-1, 1] \to [-1, 1] \ : \ x \to F(x) = 1 - \lambda x^2$$

where $0 < \lambda \leq 2$ so that $[-1, 1]$ is invariant, and

$$G : [0, 1] \to [0, 1] \ : \ x \to G(x) = \mu x(1 - x)$$

where $0 < \mu \leq 4$ so that $[0, 1]$ is invariant.

Show that these two maps are topologically conjugate by a linear function $H(x) = ax + b$ for some $a, b \in \mathbb{R}$. Provide a value $\lambda \in (0, 2]$ for which you can claim that the system $x(t + 1) = 1 - \lambda x^2(t)$ is chaotic. Conversely, provide an example of values $\mu \in (0, 4]$ for which you can claim that the system $x(t + 1) = \mu x(t)(1 - x(t))$ is not chaotic.

Exercise 3 The (left) *shift map* $S : \Omega \to \Omega$, where Ω that is the set of all binary sequences $(\omega_1, \omega_2, \omega_3 \ldots)$ of 0's ans 1's, is defined by

$$S(\omega_0, \omega_1, \omega_2, \omega_3 \ldots) = (\omega_1, \omega_2, \omega_3, \omega_4 \ldots).$$

Show that the shift map S is continuous. In other words, show that for any $\varepsilon > 0$ we can find $\delta > 0$ such that for any two binary sequences $a, b \in \Omega$, if $d(a, b) < \delta$ then $d(S(a), S(b)) < \varepsilon$, where the distance between a and b is

$$d(a, b) = \sum_{i=0}^{\infty} |a_i - b_i| \cdot 2^{-i}.$$