Dynamical Systems
for Engineers: Exercise Set 11

Exercise 1

Let $\Omega = \{1,2,3,4,5,6\}$, $A = \{1,2,3\}$, $B = \{2,4,6\}$.

1. What is the sigma-algebra Σ_A generated by A ?
2. What is the sigma-algebra Σ_B generated by B ?
3. Is $\Sigma_A \cap \Sigma_B$ a sigma-algebra ?
4. Is $\Sigma_A \cup \Sigma_B$ a sigma-algebra ?

Exercise 2

Let Ω be the interval $[0,1) = [0,1[\subset \mathbb{R}$. The Borel σ-algebra of the interval $\Omega = [0,1)$ is the smallest σ-algebra containing the intervals $[a,b], [a,b), (a,b], (a,b)$ for any $0 \leq a \leq b \leq 1$. Now, let us consider instead the collection Γ of intervals containing only all the semi-open intervals $[a,b)$ for any $0 \leq a \leq b \leq 1$, and all the finite unions that we can construct from them. Is Γ a σ-algebra ?

Exercise 3

We have seen in class that the Bernouilli map preserves the Lebesgue measure on the unit interval $\Omega = [0,1]$, which is defined by $P([a,b]) = P([a,b)) = P((a,b]) = P((a,b)) = b - a$, for any $0 \leq a \leq b \leq 1$.

Let us consider another map, called the continued fraction map, is given by

$$F(x) = \frac{1}{x} \mod 1 = \begin{cases} \frac{1}{x} - n & \text{if} \quad \frac{1}{n+1} < x \leq \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Observe that the restriction of F to an interval $(1/(n+1), 1/n]$ is monotone, with the image $F((1/(n+1), 1/n]) = [0,1)$. Observe also that the pre-image of any interval $[a,b]$ through the continued fraction map is

$$F^{-1}([a,b]) = \bigcup_{n=1}^{\infty} \left[\frac{1}{b+n}, \frac{1}{a+n}\right]. \tag{1}$$

Note that since $0 \leq a \leq b \leq 1$, the intervals $\left[\frac{1}{b+n}, \frac{1}{a+n}\right]$ are all disjoints.

Let $P(\cdot)$ be the Gauss measure, which is defined for any $0 \leq a \leq b \leq 1$ by

$$P([a,b]) = P([a,b)) = P((a,b]) = P((a,b)) = \frac{1}{\ln 2} \int_a^b \frac{1}{1+x} dx.$$

Show that the continued fraction map preserves the Gauss measure, and therefore that it is a measure preserving transformation for which the Gauss measure is invariant.