5

Stability of Nonlinear Systems

5.1 Introduction

We will distinguish two levels of stability for nonlinear systems: Large scale stability and Small scale stability.

(i) A dynamical system is globally stable (*Large scale stability*) if its solutions remain bounded.

(ii) It is locally stable (*Small scale stability*), if its solutions that start from close initial conditions remain close for all subsequent times.

We did not need to make this distinction for linear systems, because linear systems combine these kinds of stability/instability properties:

(i) Large scale stability/instability properties: Solutions of stable linear systems remain bounded whereas solutions of unstable linear systems diverge to infinity.

(ii) Small scale stability properties: Solutions of stable linear systems that start with close initial conditions remain close, whereas solutions of unstable linear systems drift apart.

In the case of nonlinear systems, small-scale stability/instability properties are not linked to the large-scale stability/instability properties.

Example: Van der Pol Oscillator

Let us consider an introductory example, which is the Van der Pol oscillator that we have seen in Section 4.2.2. It is described by (4.5) and (4.6), where we set $\lambda = 0.3$ in this example. As shown in Figure 5.1. If a solution starts exactly at the origin, it remains there forever. If, however, it starts close to the origin, it diverges from it, but without diverging to infinity. In fact it converges to a periodic solution. We have therefore a system with small scale instability but large scale stability.

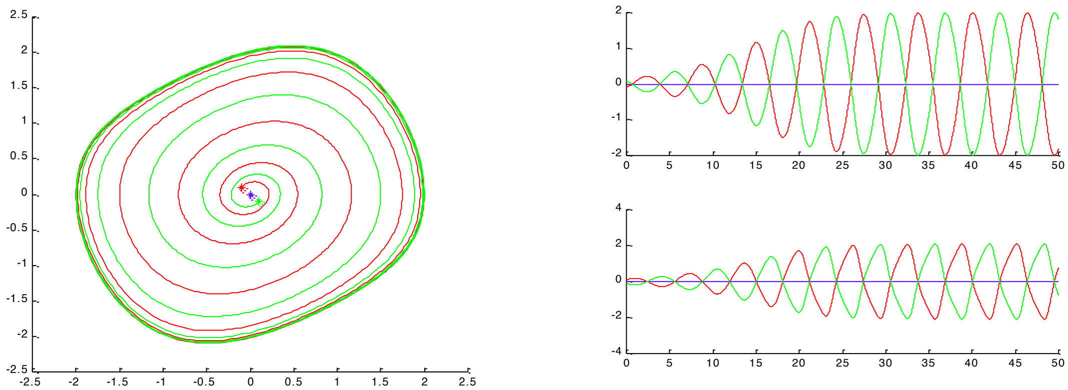

Figure 5.1: Three solutions of the Van der Pol oscillator, with $\lambda = 0.3$. Orbits (left) and trajectories as a function of time (right): The first solution (blue) remains at the origin. The two other solutions (red and green) start close to the origin, but diverge from the origin and converge to the periodic solution.

5.2 Large-scale Notions of Stability/instability

Definition 5.1 (Large-scale stability)**.** *(i) An autonomous system has* bounded solutions *if for each* $x_0 \in \mathbb{R}^n$ *there is a constant* $B > 0$ *such that for all* $t \geq 0$

$$\|x(t)\| \leq B$$

where $x(t)$ *is the solution with initial condition* $x(0) = x_0$*.*

(ii) An autonomous system has asymptotically uniformly bounded solutions *if there is a constant* $B > 0$ *such that for each* $x_0 \in \mathbb{R}^n$*, there is a finite time* $T \geq 0$ *such that for all* $t \geq T$

$$\|x(t)\| \leq B$$

where $x(t)$ *is the solution with initial condition* $x(0) = x_0$*.*

Figure 5.2 shows some typical solutions of a system with bounded and asymptotically uniformly bounded solutions. Note that an autonomous system having asymptotically uniformly bounded solutions has bounded solutions (because for the solution $x(t)$ being bounded for $t \geq T$, it must also be bounded for all $0 \leq t \leq T$). The reverse is not true, see Figure 5.2.

How is it possible to prove these properties for a specific system? We cannot hope to write down the solutions and then to check the property directly on the solutions. However, the following theorem gives a method that is most of the time either easy to apply or it is easy to see that the system has solutions that diverge to infinity. The method uses auxiliary functions, called *Lyapunov functions*, which are functions verifying the conditions in the following theorem.

Depending on the situation, a Lyapunov is either non increasing (part (i) of the theorem) or decreasing (part (ii) of the theorem) along its trajectories. For a continuous-time system $\dot{x} = F(x)$, W is *non-increasing along trajectories*, if for any solution $x(t)$,

$$\dot{W}(x) = \nabla_x^T W(x) F(x) \leq 0, \tag{5.1}$$

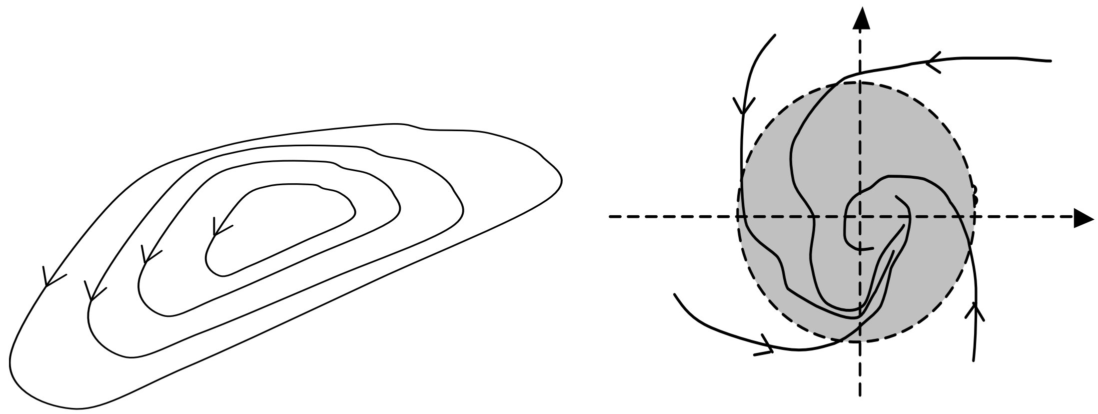

Figure 5.2: Typical orbits of a system with bounded, but not asymptotically uniformly bounded solutions (left) and of a system with asymptotically uniformly bounded solutions (right).

where T denotes transposition and where $\nabla_x W(x)$ is the gradient of W at point x:

$$\nabla_x W(x) = \begin{bmatrix} \frac{\partial W}{\partial x_1}(x_1, \ldots, x_n) \\ \frac{\partial W}{\partial x_2}(x_1, \ldots, x_n) \\ \vdots \\ \frac{\partial W}{\partial x_n}(x_1, \ldots, x_n) \end{bmatrix}.$$

For a discrete-time system $x(t+1) = F(x(t))$, (5.1) is replaced by

$$W(x(t+1)) - W(x(t)) = W(F(x(t)) - W(x(t)) \leq 0. \tag{5.2}$$

If we replace the inequality sign in (5.1) and (5.2) by a strict inequality, W is *decreasing along trajectories.*

Before stating the theorem, let us remind that the Kth-level set of a function $W : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{L}_K = \{x \in \mathbb{R}^n \mid W(x) \leq K\}$.

Theorem 5.1. *Suppose there is a function $W : \mathbb{R}^n \to \mathbb{R}$ such that*

- *for discrete time systems, W is continuous and for continuous time systems, W is continuously differentiable,*
- *$W(x) \geq 0$ for all $x \in \mathbb{R}^n$,*
- *the level sets $\mathcal{L}_K = \{x \in \mathbb{R}^n \mid W(x) \leq K\}$ are bounded for all $K > 0$.*

(i) Suppose in addition that

- *W is non-increasing along trajectories,*

then the system has bounded solutions.

(ii) Suppose in addition that there is a constant $E > 0$ such that

- *W is decreasing along trajectories as long as $W(x(t)) \geq E$ (i.e. for $x(t) \notin \mathcal{L}_E$)*

- *for a discrete-time system, the level set $\mathcal{L}_E$ is forward invariant,*

then the system has asymptotically uniformly bounded solutions.

Proof:

Let $x(t)$ be a solution, and let $E_0 = W(x(0))$.

(i) Since W is non-increasing along trajectories, $W(x(t)) \le W(x(0)) = E_0$ for all $t \ge 0$. Hence the solution $x(t)$ belongs to the level set $\mathcal{L}_{E_0}$. Now, by assumption, $\mathcal{L}_{E_0} = \{x \in \mathbb{R}^n \mid W(x) \le E_0\}$ is bounded, and therefore the solution $x(t)$ is bounded as well.

(ii) Since the level set $\mathcal{L}_E$ is bounded, there is a constant $B > 0$ such that $\|x\| \le B$ if $x \in \mathcal{L}_E$.

If $E_0 \le E$, it is sufficient to show that the solution cannot leave the level set $\mathcal{L}_E$, i.e. that the set is forward invariant. This is an explicit assumption for discrete-time systems. For a continuous time system, it is the consequence of taking W as a differentiable function. Suppose indeed, by contradiction, that $x(t)$ leaves $\mathcal{L}_E$ at some time. This implies that there are times $0 < t_1 \le t_2 < \infty$ such that $W(x(t)) \le E$ for $0 \le t \le t_1$ and $W(x(t)) > E$ for $t_1 \le t \le t_2$. Consequently, $W(x(t_1)) = E$ and $\dot{W}(x(t_1)) > 0$. But this is in contradiction with the assumption of a decreasing function along the trajectories. Hence, $x(t)$ remains in $\mathcal{L}_E$ for all times $t \ge 0$ and thus $\|x(t)\| \le B$ for all times $t \ge 0$.

If $E_0 > E$, we have to show that the solution $x(t)$ reaches $\mathcal{L}_E$ in a finite time. Once in $\mathcal{L}_E$, it remains in $\mathcal{L}_E$ for all subsequent times and it is bounded above by B by the previous argument. Define the set $\mathcal{L}_{E,E_0} = \{x \in \mathbb{R}^n \mid E \le W(x) \le E_0\}$. Since $\mathcal{L}_{E,E_0} \subseteq \mathcal{L}_{E_0}$, it is bounded, and since $W(x)$ is continuous, it is closed. In the case of a continuous time system, the continuous function $\nabla_x^T W(x)F(x)$ has a maximum within $\mathcal{L}_{E,E_0}$. Because $W(x)$ is decreasing along its trajectories, this maximum must be negative, and thus there is $\gamma < 0$ such that

$$\max_{x \in \mathcal{L}_{E,E_0}} \left\{ \nabla_x^T W(x) F(x) \right\} \le \gamma < 0.$$

Hence, as long as $x(t) \in \mathcal{L}_{E,E_0}$, we have $\dot{W}(x(t)) \le \gamma$ and thus the time that the solution $x(t)$ spends in $\mathcal{L}_{E,E_0}$ before reaching $\mathcal{L}_E$ is at most $(E - E_0)/\gamma$. In the case of a discrete time system, the argument is the same, but instead of the function $\nabla_x^T W(x)F(x)$ one has to consider the function $W(F(x)) - W(x)$.

■

Theorem 5.1 apparently only shifts the difficulty to the existence of the Lyapunov function W. Fortunately, there is a large class of systems for which it is easy to find such a function, or it is easy to see that some solutions diverge to infinity. In mechanical and electrical systems often W has the physical meaning of energy. In this case, systems with bounded, but not asymptotically uniformly bounded solutions typically have conserved energy and systems with asymptotically uniformly bounded solutions are dissipative outside of a bounded region in state space.

5.2.1 Hamiltonian Systems

Theorem 5.1 applies primarily to systems with conserved energy. We will see later the results that apply mainly to energy dissipative systems.

Most systems with conserved energy in physics are Hamiltonian systems. Their dimension is even: $n = 2r$ for some $r \in \mathbb{N}$, and they are described by a function $H : \mathbb{R}^{2r} \to \mathbb{R}$, the Hamilton function or

Hamiltonian, from which the state equation are derived as follows, for $1 \leq i \leq r$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{5.3}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{5.4}$$

The first r arguments $q_1, \ldots, q_r$ of H are called generalized coordinates and the last r arguments $p_1, \ldots, p_r$ of H are called generalized momenta. Together, they constitute the $2r$-dimensional state vector. The fact that along solutions of the system (5.3), (5.4) the value of H is constant can be seen easily by computing

$$\begin{aligned} \dot{H}\left(q_1(t), \ldots, q_r(t), p_1(t), \ldots, p_r(t)\right) &= \sum_{i=1}^{r} \frac{\partial H}{\partial q_i}\dot{q}_i + \sum_{i=1}^{r} \frac{\partial H}{\partial p_i}\dot{p}_i \\ &= -\sum_{i=1}^{r} \dot{p}_i\dot{q}_i + \sum_{i=1}^{r} \dot{q}_i\dot{p}_i = 0. \end{aligned}$$

In the conventional setting of mechanics of m point masses of weight $m_1, \ldots, m_r$ interacting through forces derived from a potential function V, the generalized coordinates q_i are the coordinates of the particles and the Hamiltonian is of the form

$$H\left(q_1, \ldots, q_r, p_1, \ldots, p_r\right) = \sum_{i=1}^{r} \frac{p_i^2}{2m_i} + V(q_1, \ldots q_r) \tag{5.5}$$

The generalized momenta p_i satisfy, according to (5.3) and (5.5),

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m_i}$$

whence $p_i = m_i dq_i/dt$, i. e. p_i is the momentum in direction of q_i and the total energy H is the sum of the kinetic energy and the potential energy.

5.2.2 Examples

Example: Double-well Potential Hamiltonian System

Consider the system described by the Hamiltonian

$$H\left(q, p\right) = \frac{p^2}{2m} + V(q) \tag{5.6}$$

where $V(q) = (q-1)^2(q+1)^2$ is a double well potential. The solutions must move along level curves of the Hamiltonian H, as shown in Figure 5.3. They are bounded.

Example: Volterra-Lotka Prey-Predator Model

If the state space Ω is a subset of $\mathbb{R}^n$, one can adapt the conditions of Theorem 5.1 accordingly, as in this example.

One of the oldest models for the evolution of two species, the second one predating on the first one, is the Volterra-Lotka model. If $x_1(t)$ represents the population size of preys at time t and $x_2(t)$ the population size of predators at time t, the model is

$$\dot{x}_1 = ax_1 - bx_1x_2 \tag{5.7}$$

$$\dot{x}_2 = cx_1x_2 - dx_2 \tag{5.8}$$

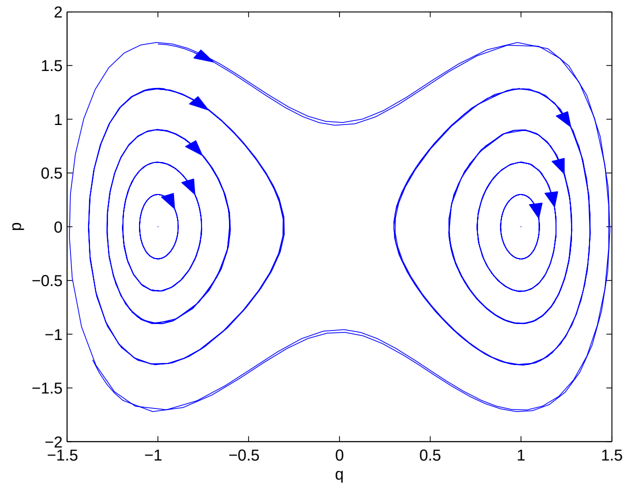

Figure 5.3: Phase portrait of the system with the Hamiltonian (5.6) in the plane (p, q). The solutions move along the level curves of the Hamiltonian H.

where $a, b, c, d > 0$. Note that the state space here is $\Omega = \mathbb{R}^2_+$. The rationale behind (5.7) is that the prey population grows exponentially at a rate a in absence of predator, but the predation reduces this growth by a rate proportional to the populations of preys and predators, with proportionality factor b. Conversely, (5.8) follows the assumptions that the predator population grows at a rate proportional to the products of populations of preys and predators, with proportionality factor c, but decays exponentially at a rate d in absence of preys on which predators can feed.

Let us assume that $b \geq a > 0$ and that $c \geq d > 0$. Let us pick

$$W(x_1, x_2) = bx_2 + cx_1 - a \ln x_2 - d \ln x_1.$$

Then $W(x_1, x_2) \geq 0$ and is continuously differentiable for all $x_1, x_2 > 0$, and in addition the level sets $\mathcal{L}_K = \{(x_1, x_2) \mid x_1 > 0, x_2 > 0, W(x) \leq K\}$ are bounded for all $K > 0$. Now, along any solution of (5.7) and (5.8), $\dot{W}(x_1, x_2) = 0$ if $x_1, x_2 > 0$, which shows that the solutions are all bounded (note that this would not be true if $x_1 = 0$). Actually, the last property of W shows that the system conserves energy: indeed it is a Hamiltonian system. To recast it in the canonical form (5.3), (5.4), make the following change of coordinates $p = \ln x_1$, $q = \ln x_2$, and $H(p, q) = W(e^p, e^q) = be^q + ce^p - aq - dp$.

The solutions $(x_1(t), x_2(t))$ oscillate around an equilibrium point $(c/d, b/a)$, as shown in Figure 5.4, showing that the model can explain some oscillatory levels of the population of preys and predator (the model was proposed by Volterra to explain the oscillations in the population of the fish species in the Adriatic).

Example: Scalar polynomial system

Consider the 1-dim. system described by

$$\dot{x} = 1 + 2x + x^2 - 3x^3. \tag{5.9}$$

To show that this system has asymptotically uniformly bounded solutions, let us pick $W(x) = x^4$. Along any solution of (5.9),

$$\dot{W}(x) = 4x^3\dot{x} = 4x^3\left(1 + 2x + x^2 - 3x^3\right) = -12x^6 + 4x^3(1 + 2x + x^2)$$

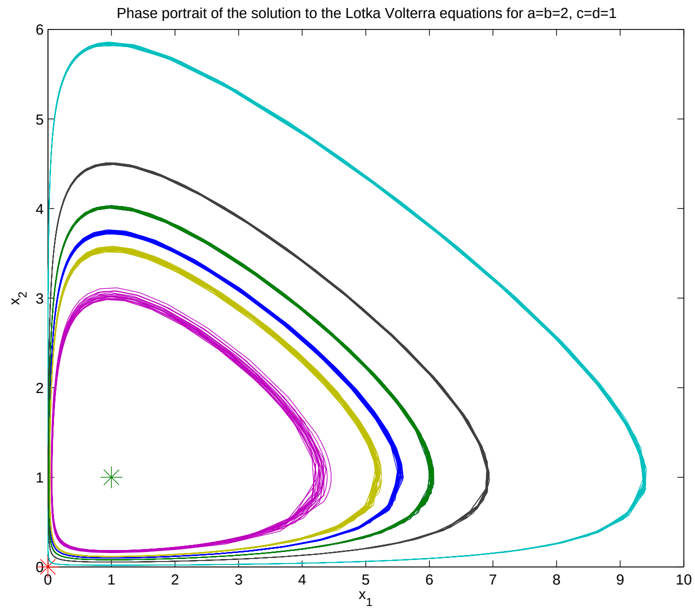

Figure 5.4: Phase portrait of the Volterra-Lotka model.

Let $C > 0$. For $|x| \geq C$, we can upper bound the right hand side of the previous equality by

$$\dot{W}(x) \leq -12x^6 + 4|x|^3 \left(\frac{|x|^3}{C^3} + 2\frac{|x|^3}{C^2} + \frac{|x|^3}{C} \right),$$

where the right hand side is now the sum of two terms in x^6. Let us pick C large enough so that the first term dominates, so that $\dot{W}(x) \leq 0$. More precisely, let us pick $C = 2$. Then

$$\dot{W}(x) \leq -15x^6/2 < 0$$

as long as $|x| \geq 2$. Furthermore, the level sets $\mathcal{L}_K = \{x \in \mathbb{R} \mid W(x) \leq K\} = [-K^{1/4}, K^{1/4}]$ are bounded for all $K > 0$. By Theorem 5.1(ii), this implies that the solutions are asymptotically uniformly bounded. Actually, the constant $C = 2$ is such an asymptotic uniform bound for the solutions.

Let us note that in this example, the sets $\{x \in \mathbb{R} \mid |x| \geq C\}$ coincide with the sets $\{x \in \mathbb{R} \mid W(x) \geq E\}$, with here $E = C^4$. In general, this is not the case, and therefore, after having proved that $\dot{W} < 0$ for $|x| \geq C$, one has to find E such that $W(x) \geq E$ implies $|x| \geq C$. Graphically, this means that the level set $\mathcal{L}_E$ of W for E is contained in the ball $\{x \in \mathbb{R} \mid |x| \geq C\}$.

Example: System with Unbounded Level Sets of Lyapunov Function

The following example shows that the condition that the level sets of W are bounded is really necessary. Consider the 1-dimensional system described by

$$\dot{x} = \frac{x}{1 + x^2}. \tag{5.10}$$

It has an equilibrium point at $x = 0$. The right hand side is positive for $x > 0$ and negative for $x < 0$. Hence, all solutions but the constant solution $x = 0$ either diverge to $+\infty$ or to $-\infty$, i.e.

the solutions are not bounded. Nevertheless, we can find a function W that is non-increasing along solutions. Indeed, let us define W by

$$W(x) = \frac{1}{1+x^2}.$$

Then

$$\frac{dW}{dt} = \frac{dW}{dx}\frac{dx}{dt} = -\frac{2x}{(1+x^2)^2}\frac{x}{1+x^2} = -\frac{2x^2}{(1+x^2)^3} \leq 0.$$

However, the level sets of W are not bounded, since

$$\mathcal{L}_K = \{x \in \mathbb{R} \mid W(x) \leq K\} = \left(-\infty, \sqrt{\frac{1-K}{K}}\right] \cup \left[\sqrt{\frac{1-K}{K}}, +\infty\right)$$

for $0 < K < 1$ and $\mathcal{L}_K = \mathbb{R}$ for $K \geq 1$.

Example: Discrete-time Recurrent Neural Network

A discrete time artificial neural network is described by the n state equations

$$x_i(t+1) = \gamma x_i(t) + \sum_{j=1}^{n} w_{ij}\sigma\left(x_j(t)\right) \tag{5.11}$$

for $1 \leq i \leq n$, where $x_i \in \mathbb{R}^n$, where $|\gamma| < 1$ and where σ is a sigmoid function, which is an odd, strictly increasing function from $\sigma(-\infty) = -1$ to $\sigma(-\infty) = 1$, with $\sigma(0) = 0$.

Let us consider the case $n = 2$; the generalization to arbitrary $n > 2$ is not difficult. Then (5.11) becomes

$$\begin{aligned} x_1(t+1) &= \gamma x_1(t) + w_{11}\sigma\left(x_1(t)\right) + w_{12}\sigma\left(x_2(t)\right) \\ x_2(t+1) &= \gamma x_2(t) + w_{21}\sigma\left(x_1(t)\right) + w_{22}\sigma\left(x_2(t)\right) \end{aligned}$$

Define

$$W(x_1, x_2) = \max\{|x_1|, |x_2|\}. \tag{5.12}$$

(i) One easily sees from (5.12) that the level sets $\mathcal{L}_K$ are bounded for all $K > 0$.

Let us next set

$$E = \frac{2a}{1-|\gamma|} \tag{5.13}$$

where $a = \max\{|w_{11}| + |w_{12}|, |w_{21}| + |w_{22}|\}$.

(ii) The level set $\mathcal{L}_E$ is forward invariant. Indeed, let $W(x_1(t), x_2(t)) \leq E$. Then it is easy to see that

$$\begin{aligned} |x_1(t+1)| &\leq |\gamma||x_1(t)| + |w_{11}| + |w_{12}| \\ &\leq |\gamma|\frac{2a}{1-|\gamma|} + a = \frac{a(1+|\gamma|)}{1-|\gamma|} \leq \frac{2a}{1-|\gamma|} \end{aligned}$$

and similarly for $|x_2(t+1)|$. Therefore $W(x_1(t+1), x_2(t+1)) \leq E$.

(iii) If $W(x_1(t), x_2(t)) \geq E$ then $W(x_1(t+1), x_2(t+1)) \leq W(x_1(t), x_2(t))$. Indeed, suppose that

$|x_1(t)| \geq |x_2(t)|$ (The opposite case is analogous). Then

$$\begin{array}{rcl} |x_1(t+1)| & \leq & |\gamma||x_1(t)| + |w_{11}| + |w_{12}| \\ & \leq & |\gamma||x_1(t)| + a \\ & = & \dfrac{1+|\gamma|}{2}|x_1(t)| - \dfrac{1-|\gamma|}{2}|x_1(t)| + a \\ & \leq & \dfrac{1+|\gamma|}{2}|x_1(t)| - \dfrac{1-|\gamma|}{2}\dfrac{2a}{1-|\gamma|} + a \\ & = & \dfrac{1+|\gamma|}{2}|x_1(t)| \\ & < & |x_1(t)| = W(x_1(t), x_2(t)) \end{array}$$

which also yields that

$$\begin{array}{rcl} |x_2(t+1)| & \leq & |\gamma||x_2(t)| + |w_{21}| + |w_{22}| \\ & \leq & |\gamma||x_2(t)| + a \\ & \leq & |\gamma||x_1(t)| + a \\ & < & W(x_1(t), x_2(t)). \end{array}$$

Therefore $W(x_1(t+1), x_2(t+1)) < W(x_1(t), x_2(t))$.

Consequently, all hypotheses of Theorem 5.1(ii) are satisfied and we conclude that the system has uniformly asymptotically bounded solutions. We also have an explicit asymptotic bound. In the norm (5.12) this bound is (5.13), whereas in the Euclidean norm it is $2\sqrt{2}a/(1-|\gamma|)$.

5.3 Small-scale Notions of Stability/instability

The local notions of stability/instability cannot be defined in any satisfactory way for a whole nonlinear system. In fact, for the Van der Pol oscillator (4.5) and (4.6), with $\lambda = 0.3$, the equilibrium point is unstable, whereas the periodic solution (limit cycle) is stable. Hence, the local stability has to refer to the single solution, and not to the system as a whole. There are various ways to define local stability. We shall limit our attention to the most important definition of stability of a solution, also called Lyapunov stability.

Definition 5.2 (Small-scale stability)**.** *(i) A solution $x^\star : \mathbb{N} \to \mathbb{R}^n$ of a discrete-time autonomous system, or a solution $x^\star : \mathbb{R}_+ \to \mathbb{R}^n$ of a continuous-time system, is* stable*, if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any solution x with*

$$\|x(0) - x^\star(0)\| \leq \delta$$

we have for all $t \geq 0$

$$\|x(t) - x^\star(t)\| \leq \varepsilon.$$

- A solution is unstable *if it is not stable.*

(ii) A solution $x^\star$ is asymptotically stable*, or more precisely locally asymptotically stable, if it is stable, i.e. if it verifies the conditions in item (i) of the definition, and if in addition*

$$\lim_{t\to+\infty} \|x(t) - x^\star(t)\| = 0. \tag{5.14}$$

The basin of attraction *at time $t = 0$ of an asymptotically stable solution $x^\star$ is the set of all $x_0 \in \mathbb{R}^n$ such that the solution x with initial condition $x(0) = x_0$ satisfies (5.14).*

(iii) A solution $x^\star$ is globally asymptotically stable *if it is asymptotically stable and if its basin of attraction is the whole space $\mathbb{R}^n$. In this case, actually all solutions are globally asymptotically stable and for any two solutions (5.14) holds. Thus, this is a property of the system rather than of the solution and the system is said to have a* unique asymptotic behavior*.*

Example: Van der Pol Oscillator

Let us consider again the Van der Pol oscillator described by (4.5) and (4.6), with $\lambda = 0.3$. From our numerical simulations in Figure 5.5 we can conjecture (and this will be shown rigorously later) that the constant solution $x_c(t) = 0$ for all $t \geq 0$ is unstable, whereas the periodic solution $x_p(t) = x_p(t+T)$ for all $t \geq 0$ is stable. It may seem that the periodic solution is actually asymptotically stable, but this is not the case, because with x_p there is an infinity of T-periodic solutions $x_{p,\tau}(t) = x_p(t-\tau)$ that differ only by a phase shift from x_p. Even if τ is very small, $x_{p,\tau}$ never converges to x_p. Thus, there is no neighborhood of the initial conditions of x_p such that all solutions starting in that neighborhood will eventually converge to x_p as required for asymptotic stability.

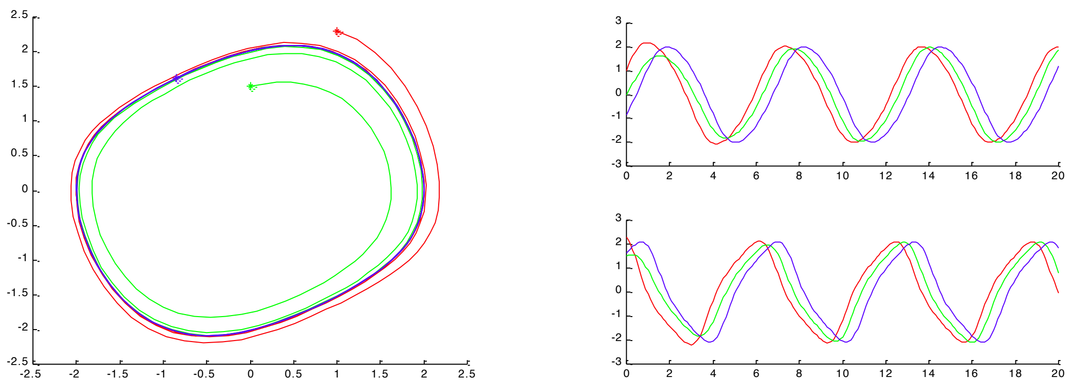

Figure 5.5: Three periodic solutions of the Van der Pol oscillator, with $\lambda = 0.3$. Orbits (left) and trajectories as a function of time (right).

By numerical simulation it appears that all solutions other than $x_c = 0$ converge to some periodic solution x_p. Thus, all solutions, except $x_c = 0$, are stable.

5.3.1 Stability of an equilibrium point in a discrete-time system

So far, the definition of local stability has been given, but no criterion how to determine in a specific case the stability or instability of a solution. There is no easy-to-use criterion for arbitrary solutions. But for fixed/equilibrium points, i.e. when $x^\star(t)$ in Definition 5.2 is a constant solution $\overline{x}$, essentially the stability question is solved by looking merely at the linearized system.

For the terminology, we do not distinguish between the fixed/equilibrium point and the constant solution remaining in the fixed/equilibrium point. Thus, e.g. a fixed/equilibrium point is stable if the constant solution starting at this fixed/equilibrium point is stable.

We start by discrete-time dynamical system. Consider the autonomous discrete time system

$$x(t+1) = F(x(t)) \tag{5.15}$$

with $F : \mathbb{R}^n \to \mathbb{R}^n$. Let $\overline{x}$ be a fixed point of F, i.e. a point such that

$$F(\overline{x}) = \overline{x}. \tag{5.16}$$

Consider a solution $x : \mathbb{N} \to \mathbb{R}^n$ that starts in a close neighborhood of $\overline{x}$. As long as the solution remains close, the following is a good approximation for the time evolution of the increment $\Delta x = x - \overline{x}$ with respect to the fixed point:

$$\begin{aligned} \Delta x(t+1) &= x(t+1) - \overline{x} \\ &= F(x(t)) - F(\overline{x}) \\ &\approx J(\overline{x})\,(x(t) - \overline{x}) \\ &= J(\overline{x})\Delta x(t), \end{aligned}$$

where $J(\overline{x})$ is the *Jacobian matrix* of F at the fixed point $\overline{x}$:

$$J(\overline{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(\overline{x}) & \frac{\partial F_1}{\partial x_2}(\overline{x}) & \dots & \frac{\partial F_1}{\partial x_n}(\overline{x}) \\ \frac{\partial F_2}{\partial x_1}(\overline{x}) & \frac{\partial F_2}{\partial x_2}(\overline{x}) & \dots & \frac{\partial F_2}{\partial x_n}(\overline{x}) \\ \vdots & & \ddots & \\ \frac{\partial F_n}{\partial x_1}(\overline{x}) & \frac{\partial F_n}{\partial x_2}(\overline{x}) & \dots & \frac{\partial F_n}{\partial x_n}(\overline{x}) \end{bmatrix} \tag{5.17}$$

The *linearized system* of the discrete time system (5.15) around the fixed point $\overline{x}$ is thus given by

$$\Delta x(t+1) = J(\overline{x})\Delta x(t). \tag{5.18}$$

Since in a neighborhood of the fixed point, the system is well approximated by its linearization, it is plausible that the stability/instability of the fixed point is usually given by the eigenvalues of $J(\overline{x})$. Indeed, the following theorem shows that this is often the case.

A fixed point $\overline{x}$ of a discrete time system is *hyperbolic*, if no eigenvalue of the Jacobian matrix at $J(\overline{x})$ lies on the unit circle (i.e. has magnitude equal to 1).

Now, we can state the stability criterion for fixed points.

Theorem 5.2. *Let $\overline{x}$ be a hyperbolic fixed point of a discrete time autonomous system given by (5.15).*

(i) The fixed point $\overline{x}$ is asymptotically stable if and only if all eigenvalues λ_i of the Jacobian matrix $J(\overline{x})$ satisfy $|\lambda_i| < 1$.

(ii) The fixed point $\overline{x}$ is unstable if and only if at least one eigenvalue λ_i of the Jacobian matrix $J(\overline{x})$ satisfies $|\lambda_i| > 1$.

As long as the fixed point is hyperbolic, the asymptotic stability of the linearized system at the fixed point and the asymptotic stability of the fixed point of the nonlinear system are equivalent. Conversely, as long as the fixed point is hyperbolic, the instability of the linearized system at the fixed point and the instability of the fixed point of the nonlinear system are equivalent. However, Theorem 5.2 gives no information for the case when the fixed point is not hyperbolic. In fact, in this case, the linearized system may have a different behavior near the fixed point as the nonlinear system.

Example: Logistic Map

Let us consider the simple 1-dim discrete-time dynamical system given by the iterations of an interval on the real line (Example 4.2.1).

$$x(t+1) = 1 - \lambda x^2(t), \tag{5.19}$$

where $0 < \lambda \leq 2$ so that the interval $[-1, 1]$ invariant. Its only fixed point in $[-1, 1]$ is

$$\overline{x} = \frac{-1 + \sqrt{1 + 4\lambda}}{2\lambda}$$

and the magnitude of the Jacobian at the fixed point is

$$|J(\overline{x})| = \left|\frac{df}{dx}(\overline{x})\right| = 2\lambda\overline{x} = -1 + \sqrt{1+4\lambda},$$

which is strictly less than 1 if and only if $0 < \lambda < 3/4$. Thus, the criterion for asymptotic stability of the fixed point is met for $0 < \lambda < 3/4$. Similarly, instability is deduced from Theorem 5.2 for $3/4 < \lambda \leq 2$. This is confirmed by the simulations of Figure 5.6.

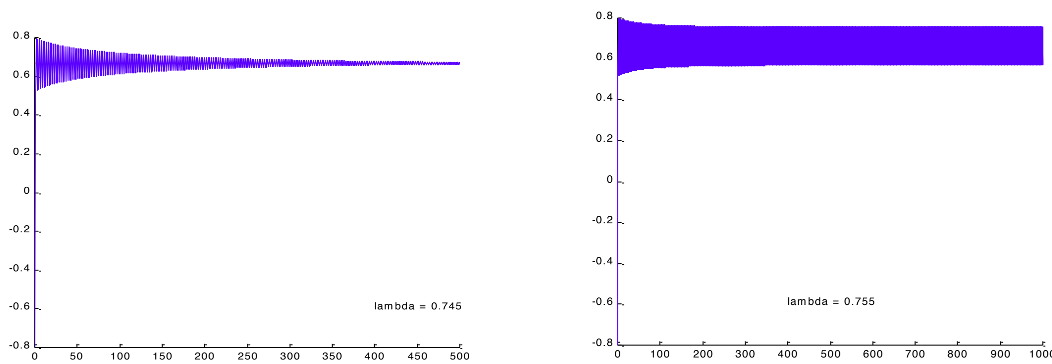

Figure 5.6: 500 iterations of the logistic map with $\lambda = 0.745$ (left) and 1000 iterations of the logistic map with $\lambda = 0.755$ (right).

At the bifurcation point $\lambda = 0.75$, i.e. at the parameter value where the behavior of the system changes qualitatively, we have

$$|J(\overline{x})| = \left|\frac{df}{dx}(\overline{x})\right| = 1.$$

Here, the fixed point is not hyperbolic and the linearization at the fixed point does not give any information on its stability. We have to take higher order approximations of the system in a neighborhood of the fixed point into account. Doing so, one can show that the fixed point is still asymptotically stable. The convergence, however, is much slower than for values of λ off the bifurcation point, as shown in Figure 5.7.

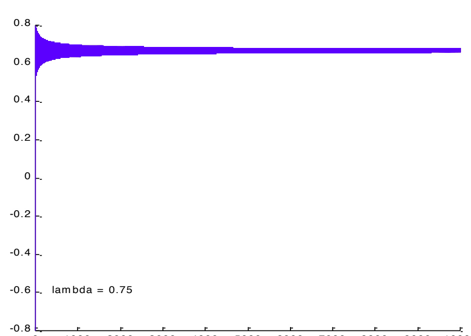

Figure 5.7: 1000 iterations of the logistic map with $\lambda = 0.75$.

5.3.2 Linearization of a continuous time system in a neighborhood of an equilibrium point

Now we consider equilibrium points of autonomous continuous time systems, Again, the linearized system allows to determine the stability or instability of the equilibrium points in most cases.

$$\frac{dx}{dt}(t) = F(x(t)) \tag{5.20}$$

with $F : \mathbb{R}^n \to \mathbb{R}^n$. Let $\overline{x}$ be an equilibrium point, i.e. a point such that

$$F(\overline{x}) = 0. \tag{5.21}$$

Consider a solution $x : \mathbb{R}_+ \to \mathbb{R}^n$ that starts in a close neighborhood of $\overline{x}$. As long as the solution remains close, the following is a good approximation for the time evolution of the increment $\Delta x = x - \overline{x}$ with respect to the fixed point:

$$\begin{aligned} \frac{d\Delta x}{dt}(t) &= \frac{dx}{dt}(t) - \frac{d\overline{x}}{dt}(t) = \frac{dx}{dt}(t) \\ &= F(x(t)) - F(\overline{x}) \\ &\approx J(\overline{x})\,(x(t) - \overline{x}) \\ &= J(\overline{x})\Delta x(t), \end{aligned}$$

where $J(\overline{x})$ is the *Jacobian matrix* of F at the fixed point $\overline{x}$, given by (5.17). The *linearized system* of the continuous time system (5.20) around the fixed point $\overline{x}$ is thus given by

$$\frac{d\Delta x}{dt}(t) = J(\overline{x})\Delta x(t) \tag{5.22}$$

Since in a neighborhood of the fixed point, the system is well approximated by its linearization, it is again plausible that the stability/instability of the fixed point is usually given by the eigenvalues of $J(\overline{x})$. Indeed, the following theorem shows that this is often the case.

An equilibrium point $\overline{x}$ of a continuous-time system is *hyperbolic*, if no eigenvalue of the Jacobian matrix at $J(\overline{x})$ lies on the imaginary axis (i.e. has a zero real part).

Now, we can state the stability criterion for equilibrium points.

Theorem 5.3. *Let $\overline{x}$ be a hyperbolic equilibrium point of a continuous-time autonomous system given by (5.20).*

(i) The equilibrium point $\overline{x}$ is asymptotically stable if and only if all eigenvalues λ_i of the Jacobian matrix $J(\overline{x})$ satisfy $\Re\{\lambda_i\} < 0$.

(ii) The equilibrium point $\overline{x}$ is unstable if and only if at least one eigenvalue λ_i of the Jacobian matrix $J(\overline{x})$ satisfies $\Re\{\lambda_i\} > 0$.

We make the same observation as for discrete-time systems with Theorem 5.2: As long as the fixed point is hyperbolic, the asymptotic stability of the linearized system at the fixed point and the asymptotic stability of the fixed point of the nonlinear system are equivalent. Conversely, as long as the fixed point is hyperbolic, the instability of the linearized system at the fixed point and the instability of the fixed point of the nonlinear system are equivalent. However, Theorem 5.3 gives no information for the case when the fixed point is not hyperbolic. In fact, in this case, the linearized system may have a different behavior near the fixed point as the nonlinear system.

Example: Van der Pol Oscillator

Consider again the Van der Pol oscillator (4.5) and (4.6). It has the equilibrium point $\overline{x} = 0$. Its Jacobian matrix is

$$J(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -1 - 2\lambda x_1 x_2 & -\lambda(x_1^2 - 1) \end{bmatrix}$$

and therefore at the equilibrium point

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & \lambda \end{bmatrix}$$

whose eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(\lambda \pm \sqrt{\lambda^2 - 4}\right).$$

The real part of the eigenvalues is $\lambda/2$. Consequently, Theorem 5.3 yields that the origin is an asymptotically stable equilibrium point if $\lambda < 0$ and an unstable equilibrium point if $\lambda > 0$. This is confirmed by numerical simulations of Figure 5.8.

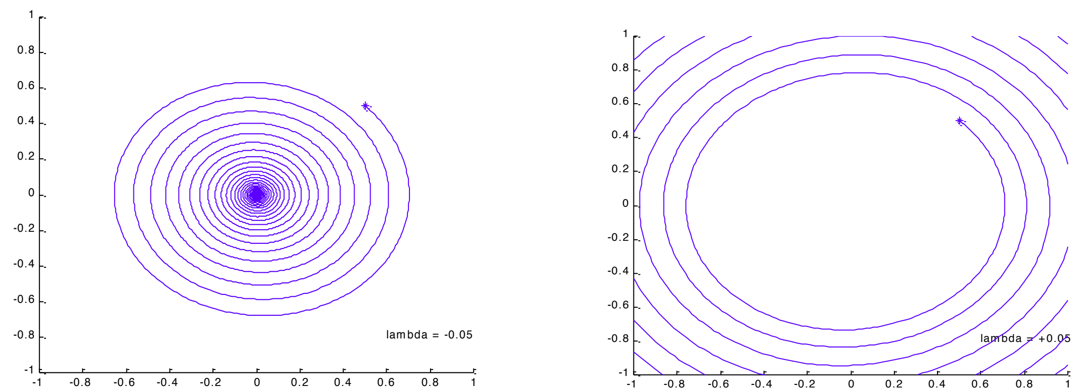

Figure 5.8: Orbit of a solution of the Van der Pol oscillator, with $\lambda = -0.05$ (left) and $\lambda = 0.05$ (right).

At the bifurcation point $\lambda = 0$, i.e. at the parameter value where the behavior of the system changes qualitatively, both eigenvalues are purely imaginary. The equilibrium point is not hyperbolic and the linearization at the fixed point does not give any information on its stability. Equation (4.6) is however linear for this value of λ, and therefore the solutions are sinusoidal. The amplitude and the phase of the oscillation depend on the initial condition, as shown in Figure 5.9. The solutions are stable, but not asymptotically stable: one can check that the origin is a center. In this special case, the system and its linearization around the equilibrium point coincide and therefore the stability properties of the linearization and of the system are identical.

5.3.3 Nature of the flow in a neighborhood of an equilibrium/fixed point

If a fixed/equilibrium point is hyperbolic, not only the stability of the fixed/equilibrium point of the nonlinear system is the same as for the corresponding linearized system, but the whole flow of the nonlinear system and the linearized system are equivalent in a neighborhood of the fixed/equilibrium point. We will not introduce here a rigorous notion of equivalence, but in equivalent flows, the solutions correspond to each other in a bijective fashion and they have the same convergence properties.

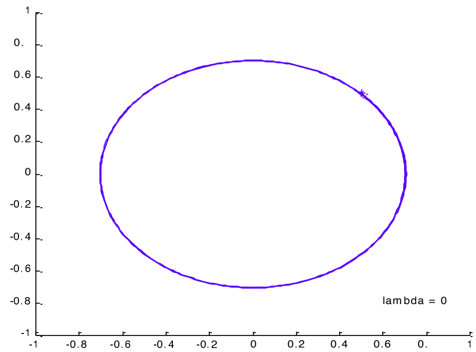

Figure 5.9: Orbit of a solution of the Van der Pol oscillator, with $\lambda = 0$ and initial conditions $(x_1(0), x_2(0)) = (0.5, 0.5)$.

Remember that for a linear system, with a equilibrium/fixed point at the origin ($x = 0$), any arbitrary initial condition x_0 gives rise to a solution $x(t)$ which can be decomposed into

- a contracting linear subspaceV_s, which is the subspace spanned by the eigenvectors with eigenvalues lying in the interior of the unit circle/left half-plane, and
- an expanding linear subspace V_u, which is the subspace spanned by the eigenvectors with eigenvalues lying in the exterior of the unit circle/interior of the right half-plane.

For a nonlinear system, the notions of eigenspaces are extended as follows.

Definition 5.3. *The* stable manifold W^s *of an equilibrium/fixed point* $\overline{x}$ *is the set*

$$W^s = \left\{ x_0 \mid x(t) \text{ is a solution starting at } x(0) = x_0 \text{ such that } \lim_{t \to \infty} x(t) = \overline{x} \right\}.$$

The unstable manifold W^u *of an equilibrium/fixed point* $\overline{x}$ *is the set*

$$W^u = \left\{ x_0 \mid x(t) \text{ is a solution starting at } x(0) = x_0 \text{ such that } \lim_{t \to -\infty} x(t) = \overline{x} \right\}.$$

Let $\overline{x}$ be an equilibrium/fixed point and V^s (respectively, V^u) be the contracting (resp., expanding) linear subspace of the linearization of the system at $\overline{x}$. The manifolds of $\overline{x}$ enjoy the following properties:

- The stable manifold W^s of $\overline{x}$ is an invariant set. In a neighborhood of $\overline{x}$, W^s is a surface of the same dimension as V^s and it is tangent to V^s at $\overline{x}$.
- The unstable manifold W^u of $\overline{x}$ is an invariant set. In a neighborhood of $\overline{x}$, W^u is a surface of the same dimension as V^u and it is tangent to V^u at $\overline{x}$.

This property explains the qualitative behavior of the flow in a neighborhood of an equilibrium/fixed point. For instance, in the case of a continuous-time 2-dimensional system with an equilibrium/fixed point where the Jacobian matrix has one positive and one negative eigenvalue, the stable and unstable manifold are 1-dimensional and their portrait is sketched in Figure 5.10 (left). The structure of the flow of the system then follows similarly to that of a saddle node in a linear system.

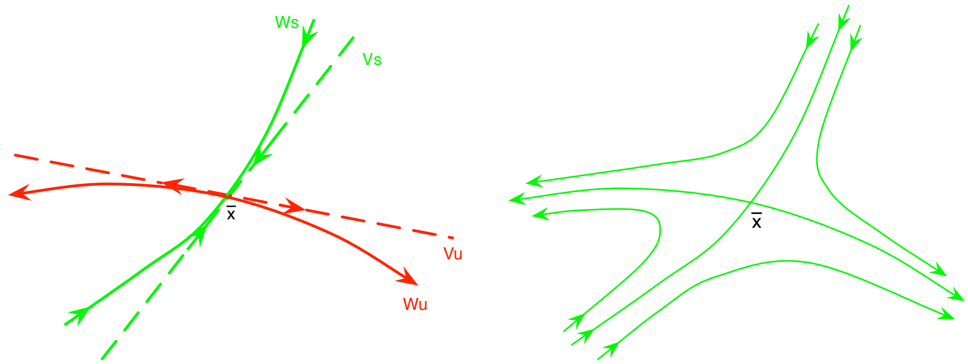

Figure 5.10: Stable and unstable manifold of a hyperbolic equilibrium point, which are tangent to the contracting (resp., expanding) linear subspace of the linearized system (left), and phase portrait in the neighborhood of the equilibrium point

5.3.4 Estimation of the basins of attraction of asymptotically stable fixed/equilibrium points

When an equilibrium/fixed point is asymptotically stable, one would like to have an idea of the size of its basin of attraction. For this purpose, we again resort to Lyapunov functions. Before stating the theorem, let us introduce the strict Kth-level set of a function $W : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{U}_K = \{x \in \mathbb{R}^n \mid W(x) < K\}$.

Theorem 5.4. *Let $\overline{x}$ be an equilibrium/fixed point. Suppose there is a function $W : \mathbb{R}^n \to \mathbb{R}$ and a constant $E > 0$ such that*

- *for discrete time systems, W is continuous and for continuous time systems, W is continuously differentiable,*
- *$W(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{\overline{x}\}$, and $W(\overline{x}) = 0$,*
- *the strict level set $\mathcal{U}_E = \{x \in \mathbb{R}^n \mid W(x) < E\}$ is bounded,*
- *W is non-increasing along trajectories as long as $W(x(t)) < E$ (i.e. for $x(t) \in \mathcal{U}_E$),*
- *W is decreasing along trajectories as long as $W(x(t)) < E$ and $x(t) \neq \overline{x}$ (i.e. for $x(t) \in \mathcal{U}_E \setminus \{\overline{x}\}$).*

Then U_E is contained in the basin of attraction of $\overline{x}$.

Proof:

Let $x(t)$ be a solution with $W(x(0)) < E$, i.e. with $x(0) \in \mathcal{U}_E$.

Since W is non-increasing along trajectories as long as $W(x(t)) < E$, it follows that $x(t) \in \mathcal{U}_E$ for all $t \geq 0$. Since $\mathcal{U}_E$ is bounded, $x(t)$ is bounded for all $t \geq 0$. Let $\mathcal{S}_\omega$ be the ω-limit set of the solution $x(t)$. Since W is decreasing along trajectories as long as $W(x(t)) < E$ and $x(t) \neq \overline{x}$ while being from below by 0, there must be a constant $E_\infty \geq 0$ such that $W(t) \to E_\infty$ when $t \to +\infty$. Therefore, for any $y \in \mathcal{S}_\omega$, $W(y) = E_\infty$. Consider an arbitrary state $y_0 \in \mathcal{S}_\omega$. Because the solution $x(t)$ is bounded, its ω-limit set $\mathcal{S}_\omega$ is invariant, hence the whole solution $y(t)$ starting at $y(0) = y_0$ lies in $\mathcal{S}_\omega$, and leads to constant value of W: $W(y(t)) = E_\infty$, for all $t \geq 0$. The last condition implies that the only solution such that W can remain constant is the equilibrium/fixed point $\overline{x}$. Therefore $y(t) = \overline{x}$. Consequently, $\mathcal{S}_\omega = \{\overline{x}\}$ and $\lim_{t\to\infty} x(t) = \overline{x}$.

■

Example: Van der Pol Oscillator

Consider again the Van der Pol oscillator (4.5) and (4.6), with $\lambda < 0$. Remember that it has the equilibrium point $\overline{x} = 0$.

Let us try $W(x_1, x_2) = x_1^2 + x_2^2$ as candidate for a Lyapunov function. We compute that

$$\dot{W}(x_1, x_2) = 2\lambda \left(1 - x_1^2\right) x_2^2 \leq 0$$

if $x_1^2 < 1$ and thus if $W(x_1, x_2) < 1$. Moreover, $\dot{W}(x_1, x_2) = 0$ with $W(x_1, x_2) < 1$ if and only if $x_2 = 0$. Since $F(x_1, 0) = (0, -x_1)$, we see that any solution starting at a initial state $(x_1(0), 0)$ on the x_1-axis leaves this axis immediately, unless $x_1 = 0$. Therefore the only value at which W is not strictly decreasing along its trajectories in the open disk $\mathcal{U}_1 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$ is the equilibrium point $\overline{x} = 0$. All conditions of the theorem are met (the other ones are immediate to verify), and we can conclude the open disk $\mathcal{U}_1 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$ is contained in the basin of attraction of $\overline{x} = 0$ for any $\lambda < 0$.

5.3.5 Globally asymptotical stability

Theorem 5.4 does not cover directly the case of a globally asymptotically stable equilibrium/fixed point, because setting $E = +\infty$ in the theorem would be in contradiction with the condition that $\mathcal{U}_E$ is bounded ($\mathcal{U}_E = \mathbb{R}^n$ if $E = +\infty$). However, each solution starts at an initial state with a finite value of W and if subsequently W always decreases, it must converge to the equilibrium/fixed point. Therefore, we can reformulate the theorem for this case, where the basin of attraction of the equilibrium/fixed point is $\mathbb{R}^n$.

Corollary 5.1. *Let $\overline{x}$ be an equilibrium/fixed point. Suppose there is a function $W : \mathbb{R}^n \to \mathbb{R}$ such that*

- *for discrete time systems, W is continuous and for continuous time systems, W is continuously differentiable,*
- *$W(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{\overline{x}\}$, and $W(\overline{x}) = 0$,*
- *the strict level sets $\mathcal{U}_K = \{x \in \mathbb{R}^n \mid W(x) < K\}$ are bounded for all $K > 0$,*
- *W is non-increasing along trajectories,*
- *W is decreasing along trajectories as long as $x(t) \neq \overline{x}$ (i.e. for $x(t) \in \mathbb{R}^n \setminus \{\overline{x}\}$).*

Then $\overline{x}$ is globally asymptotically stable.

Example: Logistic Map

Let us consider again the 1-dim discrete-time dynamical system given by (5.19), with $0 < \lambda \leq 2$ and $-1 \leq x \leq 1$ (Remember that the interval $[-1, 1]$ invariant for these values of λ). Let

$$W(x) = (x - \overline{x})^2$$

where $\overline{x} = (-1 + \sqrt{1 + 4\lambda})/(2\lambda)$ is the (only) fixed point of the system in $[-1, 1]$. Clearly, W is continuously differentiable, $W(x) > 0$ for all $x \in [-1, 1] \setminus \{\overline{x}\}$ and $W(\overline{x}) = 0$, and $\mathcal{U}_K$ is bounded for any $K > 0$. Next,

$$\begin{aligned} W(F(x)) &= (F(x) - \overline{x})^2 = (F(x) - F(\overline{x}))^2 \\ &= \left(\int_{\overline{x}}^{x} \frac{dF}{d\xi}(\xi) d\xi \right)^2 \\ &\leq \left(\int_{\overline{x}}^{x} \left| \frac{dF}{d\xi}(\xi) \right| d\xi \right)^2 . \end{aligned}$$

If $|dF/dx(x)| < 1$ for $-1 \leq x \leq 1$, then

$$W(F(x)) < \left(\int_{\overline{x}}^{x} d\xi \right)^2 = (x - \overline{x})^2 = W(x)$$

which shows that W is non-increasing along trajectories, and moreover is decreasing along trajectories as long as $x(t) \neq \overline{x}$. Since here $F(x) = 1 - \lambda x^2$, we find that

$$\left| \frac{dF}{dx}(x) \right| = 2|\lambda x| \leq 2|\lambda|$$

for $1 \leq x \leq 1$, so that $\overline{x}$ is globally asymptotically stable if $0 < \lambda < 1/2$.

If we compare this result with the result of Section 5.3.1, that the fixed point is asymptotically stable for $0 < \lambda < 3/4$, the question arises what the system behavior is for $1/2 < \lambda < 3/4$. It turns out that even for these values of λ the fixed point is globally asymptotically stable.

This is a quite common situation, when one applies the Lyapunov function method. It allows getting explicit stability results without much effort, but these results often are rather conservative in the sense that stability holds for a considerably larger range of parameters than the simple application of the method allowed to obtain. Of course, we did not explore all possible Lyapunov functions, but just the simplest one. Actually, a trick would allow to prove global asymptotic stability for the whole range $0 < \lambda < 3/4$ in this system, namely to apply the same Lyapunov function to $G(x) = F(F(x))$.

It is also instructive to compare the two conditions for (local) asymptotic stability, which is that $|dF(\overline{x})/dx| < 1$, with that of global asymptotic stability, which is that $|dF(x)/dx| < 1$ for $-1 \leq x \leq 1$. While the former imposes the condition on the derivative at the fixed point $\overline{x}$, the latter imposes the same condition on the whole interval $[-1, 1]$. Accordingly, the asymptotic stability of the fixed point that is deduced is local in the former and global in the latter.

5.3.6 Gradient Systems

There is an important class of continuous-time systems that have a natural Lyapunov function that decreases everywhere along the solutions. In fact, the Lyapunov function is used to construct the system itself.

Definition 5.4 (Gradient System)**.** *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function. Then the gradient system generated by V is defined by the state equation*

$$\dot{x} = -\nabla_x V(x). \tag{5.23}$$

It follows from this definition that along any solution $x(t)$ of the system,

$$\begin{aligned} \dot{V}(x) &= \nabla_x^T V(x) \dot{x} \\ &= \nabla T_x V(x) \left(-\nabla_x V(x) \right) \\ &= - \left\| \nabla_x V(x) \right\|^2 \\ &\leq 0. \end{aligned}$$

Therefore V is non-increasing along trajectories, yielding the following properties.

- If $\overline{x}$ is the unique equilibrium point of the system, if $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and if all strict level sets of V are bounded, then Corollary 5.1 yields that $\overline{x}$ is globally asymptotically stable: all solutions converge to $\overline{x}$.
- If $\xi(t)$ is a periodic solution, then V must be constant along $\xi(t)$. Hence

$$0 = \dot{V}(\xi(t)) = -\left\|\nabla_x V(\xi(t))\right\|^2,$$

and therefore $\nabla_x V(\xi(t)) = 0$ which implies that $\xi(t)$ is a constant solution. Gradient systems have thus no periodic solutions other than equilibrium points.
- If all equilibrium points are isolated, and if all strict level sets of V are bounded, then all solutions converge to an equilibrium point (stable or unstable, possibly different equilibrium points for different solutions). This can be seen as follows. Consider any solution. Since all level sets of V are bounded and since V is non increasing along solutions, the solution is bounded. Hence, its ω-limit set $\mathcal{S}_\omega$ is non-empty, compact, invariant and connected. Furthermore, V must be constant on $\mathcal{S}_\omega$ and by the same reasoning as for the previous item point can only contain equilibrium points. Since by hypothesis, the equilibrium points are isolated, $\mathcal{S}_\omega$ contains a single equilibrium point, towards which the solution converges.

5.3.7 Stability of a periodic solution of a discrete-time system

We now turn our attention to the stability of periodic solution. In other words, $x^\star(t)$ in Definition 5.2 is a now a T-periodic solution ξ, which is therefore such that

$$\xi(t+T) = \xi(t) \quad \text{for all } \ t \in \mathcal{T}. \tag{5.24}$$

We start with discrete-time systems ($\mathcal{T} = \mathbb{N}$), where the discussion of the stability of periodic solutions boils down to the discussion of the stability of fixed points, as shown by the following theorem.

Theorem 5.5. *Let ξ be a T-periodic solution of a discrete-time autonomous system given by (5.15), with $T \in \mathbb{N}^*$. Then $\xi(0), \xi(1), \ldots \xi(T-1)$ are fixed points of the mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$G(x) = F(F(F(\ldots(F(x))))) = F^{(T)}(x). \tag{5.25}$$

Conversely, to any fixed point of G corresponds a T-periodic solution of the system given by the iterations of F. The stability properties of the T-periodic solution ξ are the same as the stability properties of the corresponding fixed points of (5.25).

Example: Logistic Map

A 2-periodic solution of the 1-dim discrete-time dynamical system given by (5.19) corresponds to a fixed point of

$$G(x) = F(F(x)) = 1 - \lambda\left(1 - \lambda x^2\right)^2 = 1 - \lambda + 2\lambda^2 x^2 - \lambda^3 x^4,$$

or, equivalently, to a zero of the polynomial

$$P(x) = \lambda^3 x^4 - 2\lambda^2 x^2 + x - 1 + \lambda.$$

Because any fixed point of F is also a fixed point of G, and because the fixed points of F are the roots of the polynomial

$$Q(x) = \lambda x^2 + x - 1,$$

this latter polynomial must be a factor of $P(x)$, and indeed we can write

$$P(x) = Q(x)\left(\lambda^2 x^2 - \lambda x - \lambda + 1\right).$$

Therefore the 2-periodic solution is given by the zeros of $\lambda^2 x^2 - \lambda x - \lambda + 1 = 0$, which are

$$\xi_{\pm} = \frac{1 \pm \sqrt{4\lambda - 3}}{2\lambda},$$

and which are real and distinct if $\lambda > 3/4$. Moreover, this periodic solution is asymptotically stable if

$$\begin{aligned}
\left|\frac{dG}{dx}(\xi_+)\right| &= 4\lambda^2 \left|\xi_+ \left(1 - \lambda\xi_+^2\right)\right| = 4\lambda^2 \left|\xi_+\xi_-\right| = 4\left|1-\lambda\right| < 1 \\
\left|\frac{dG}{dx}(\xi_-)\right| &= 4\lambda^2 \left|\xi_- \left(1 - \lambda\xi_-^2\right)\right| = 4\lambda^2 \left|\xi_-\xi_+\right| = 4\left|1-\lambda\right| < 1,
\end{aligned}$$

which is the case for $3/4 < \lambda < 5/4$.

Variational equation

The stability of a T-periodic solution of $x = F(x)$ being equivalent to the stability of the fixed point of the system $x = F^{(T)}(x)$, we can expect that the linearization of the latter system describes well the behavior of the solutions that start close to an asymptotically stable T-periodic solution of $x = F(x)$. This is indeed done using the concept of variational equations, which describe the time evolution of the dependence of a solution on the initial conditions. For this purpose we use the notation $\Phi(t, x_0)$ for the family of all solutions for all possible initial conditions x_0, also called the flow the system, whereas we continue to use $x(t)$ for a single solution, i.e. a particular member of the family starting at a given initial condition x_0.

Let $x^\star(t)$ be a particular solution of a discrete-time autonomous system given by (5.15), with initial condition $x^\star(0)$ (Here we consider a T-periodic solution, but the variational equation can be written with respect to any such solution). If a solution x starts at an initial condition $x(0)$ close to $x^\star(0)$, and if we define its increment with respect to the solution $x^\star$ by

$$\Delta x = x - x^\star, \tag{5.26}$$

then

$$\Delta x(t) = \Phi(t, x(0)) - \Phi(t, x^\star(0)). \tag{5.27}$$

The first order approximation to the increment reads

$$\Delta x(t) = M(t)\Delta x(0) \tag{5.28}$$

where $M(t)$ is the Jacobian matrix of Φ with respect to x_0 at the point $x^\star(0)$ and for a given time t, which reads

$$M(t) = \begin{bmatrix} \frac{\partial \Phi_1}{\partial x_{01}}(t, x^\star(0)) & \frac{\partial \Phi_1}{\partial x_{02}}(t, x^\star(0)) & \dots & \frac{\partial \Phi_1}{\partial x_{0n}}(t, x^\star(0)) \\ \frac{\partial \Phi_2}{\partial x_{01}}(t, x^\star(0)) & \frac{\partial \Phi_2}{\partial x_{02}}(t, x^\star(0)) & \dots & \frac{\partial \Phi_2}{\partial x_{0n}}(t, x^\star(0)) \\ \vdots & & \ddots & \\ \frac{\partial \Phi_n}{\partial x_{01}}(t, x^\star(0)) & \frac{\partial \Phi_n}{\partial x_{02}}(t, x^\star(0)) & \dots & \frac{\partial \Phi_n}{\partial x_{0n}}(t, x^\star(0)) \end{bmatrix}. \tag{5.29}$$

Observe that $M(0) = I_n$. Now, because of (5.15), the flow $\Phi(t, x_0)$ verifies

$$\Phi(t+1, x_0) = F(\Phi(t, x_0)).$$

If we differentiate this equation with respect to x_0 at $x^\star(0)$, we obtain

$$M(t+1) = J\left(x^\star(t)\right) M(t) \tag{5.30}$$

where $J(x^\star(t))$ is the *Jacobian matrix* of F at $x^\star(t)$, given by

$$J\left(x^\star(t)\right) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(x^\star(t)) & \frac{\partial F_1}{\partial x_2}(x^\star(t)) & \dots & \frac{\partial F_1}{\partial x_n}(x^\star(t)) \\ \frac{\partial F_2}{\partial x_1}(x^\star(t)) & \frac{\partial F_2}{\partial x_2}(x^\star(t)) & \dots & \frac{\partial F_2}{\partial x_n}(x^\star(t)) \\ \vdots & & \ddots & \\ \frac{\partial F_n}{\partial x_1}(x^\star(t)) & \frac{\partial F_n}{\partial x_2}(x^\star(t)) & \dots & \frac{\partial F_n}{\partial x_n}(x^\star(t)) \end{bmatrix}$$

Equation (5.30) is called the *variational equation* of the discrete-time system along the solution $x^\star(t)$. Its solution is, since $M(0)$ is the identity matrix of order n,

$$M(t) = J\left(x^\star(t-1)\right) J\left(x^\star(t-2)\right) \cdots J\left(x^\star(1)\right) J\left(x^\star(0)\right). \tag{5.31}$$

The variational equation yields following corollary of Theorem 5.5.

Corollary 5.2. *Let ξ be a T-periodic solution of a discrete-time autonomous system given by (5.15), with $T \in \mathbb{N}^*$, such that the fixed point $\xi(0)$ of the mapping $G = F^{(T)}$ is hyperbolic. Then the solution is asymptotically stable if and only if the matrix $M(T)$ given by (5.31), with $x^\star = \xi$, has all its eigenvalues within the unit circle.*

Proof:

We use the notational shorthands

$$\frac{\partial G}{\partial x}$$

to denote the $n \times n$ matrix of partial derivatives of the n-dim function G with respect to the n-dim vector x. Because of the chain rule,

$$\begin{aligned} \frac{\partial F^{(T)}}{\partial x}(\xi(0)) &= \frac{\partial F}{\partial x}\left(F^{(T-1)}(\xi(0))\right) \frac{\partial F}{\partial x}\left(F^{(T-2)}(\xi(0))\right) \cdots \frac{\partial F}{\partial x}\left(F(\xi(0))\right) \frac{\partial F}{\partial x}\left(\xi(0)\right) \\ &= \frac{\partial F}{\partial x}\left(\xi(T-1)\right) \frac{\partial F}{\partial x}\left(\xi(T-2)\right) \cdots \frac{\partial F}{\partial x}\left(\xi(1)\right) \frac{\partial F}{\partial x}\left(\xi(0)\right) \\ &= J(\xi(t-1)) J(\xi(t-2)) \cdots J(\xi(1)) J(\xi(0)) = M(T). \end{aligned}$$

■

5.3.8 Stability of a periodic solution of a continuous-time system

Variational equation

The variational equations for continuous-time systems are obtained in a similar way. Let $x^\star(t)$ be a solution of an with initial condition $x^\star(0)$. In continuous-time, the flow $\Phi(t, x_0)$ verifies

$$\frac{\partial \Phi}{\partial t}(t, x_0) = F(\Phi(t, x_0)).$$

If we differentiate this equation with respect to x_0 at $x^\star(0)$, we obtain the variational equation for the continuous time system along the solution $x^\star$

$$\dot{M}(t) = J\left(x^\star(t)\right) M(t) \tag{5.32}$$

This is a linear time-dependent differential equation for the matrix function $M(t)$. The time dependence is caused by the argument $x^\star(t)$ of the Jacobian matrix J of F. If we combine (5.32) with the original system equation (5.20), we obtain the time-independent nonlinear system of $n+n^2$ differential equations

$$\dot{x}^\star = F(x^\star) \tag{5.33}$$
$$\dot{M} = J(x^\star)M. \tag{5.34}$$

with the initial conditions $x(0)$ and $M(0) = I_n$. For numerical calculations, it is advantageous to solve this combined system, rather than the original and the variational equations sequentially.

The matrix $M(t)$ associated with a solution $x^\star(t)$ describes the behavior of the system in a neighborhood of that solution. If we change the initial condition from $x^\star(0)$ to $x(0) = x^\star(0) + \Delta x(0)$, then the resulting solution can be approximated to the first order by

$$x(t) = x^\star(t) + \Delta x(t) \approx x^\star(t) + M(t)\Delta x(0). \tag{5.35}$$

$M(t)$ transforms the difference between the initial conditions of two solutions into another vector, which is the difference of the same solutions at time t, as a first order approximation. The better the approximation, the closer the two solutions. Often, the solutions drift apart and the approximation worsens as time goes on. To illustrate this, two solutions of the Van der Pol oscillator as well as the liner approximation of their difference are simulated in Figure 5.11.

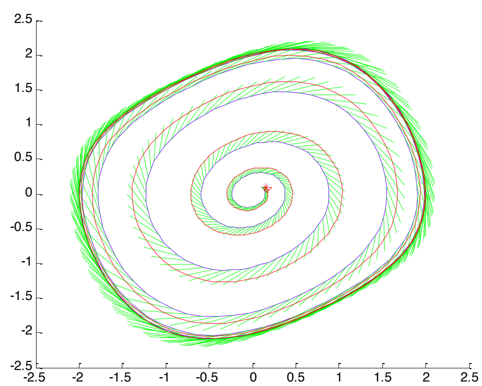

Figure 5.11: Two solutions of the Van der Pol oscillator, starting close together and close to the unstable equilibrium point at the origin. The small line segments indicate the approximation of the difference between the two solutions obtained by solving the variational equations. In the beginning, the approximation is very good, but later on it becomes less and less accurate.

Stability of a periodic solution

Suppose that the system (5.20) has a T-periodic solution ξ that verifies therefore (5.24). Let us denote by Γ the orbit of the periodic solution, which is also called a *cycle*.

$$\Gamma = \{x \in \mathbb{R}^n \mid x = \xi(t) \text{ for some } 0 \le t \le T\}.$$

Now consider the hyperplane $\mathcal{P}$ in $\mathbb{R}^n$, which intersects Γ perpendicularly at the point $\xi(0)$, i.e., which is orthogonal to the vector $\dot{\xi}(0)$:

$$\mathcal{P} = \left\{x \in \mathbb{R}^n \mid \dot{\xi}^T(0)(x - \xi(0)) = 0\right\}.$$

(Note that the exponent T in the above expression denotes transposition). This plane $\mathcal{P}$ is called the Poincaré section at $\xi(0)$.

In a neighborhood $\mathcal{U}$ of $\xi(0)$ on $\mathcal{P}$ we define the first return map R, or *Poincaré map*, as follows. Let $x_0 \in \mathcal{U}$ and let $x(t)$ be the solution with $x(0) = x_0$. This solution will intersect $\mathcal{P}$ again, after approximately time T, at a point $R(x_0)$, as shown in Figure 5.12. Clearly, $R(\xi(0)) = \xi(0)$, i.e. $\xi(0)$ is a fixed point of the return map. The following theorem is given without proof.

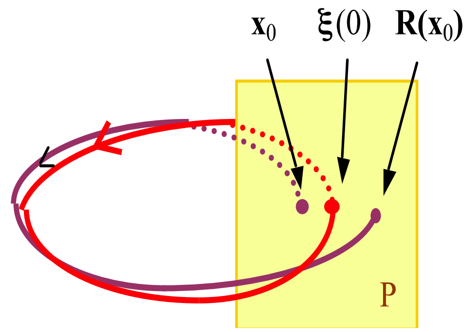

Figure 5.12: Poincaré map

Theorem 5.6. *Let ξ be a T-periodic solution of a continuous-time autonomous system given by (5.20), with $T \in \mathbb{R}^+$, and let Γ be its orbit. Consider the first return map R of the Poincaré section $\mathcal{P}$ through $\xi(0)$. Then the periodic solution ξ is stable if the fixed point $\xi(0)$ of the first return map R is stable. Furthermore, the ω-limit set of any solution x of the system starting sufficiently close to $\xi(0)$ is Γ.*

In other words, the orbit of any solution that starts close to the cycle Γ converges to the cycle Γ. One cannot conclude from this property however that the periodic solution ξ is asymptotically stable. The fixed point $\xi(0)$ of the return map R is indeed asymptotically stable, but for variations of the initial condition in the direction orthogonal to the Poincaré section, the corresponding solutions do not converge to $\xi(t)$ as $t \to +\infty$. In particular, if we choose an initial condition $x_\tau(0) = \xi(\tau)$ for some $0 < \tau < T$, then the corresponding solution is $x_\tau(t) = \xi(t + \tau)$, and even for arbitrarily small $\tau > 0$ the solution $x_\tau(t)$ never converges to $\xi(t)$. We have already mentioned this subtle difference between asymptotic stability of a periodic solution, and convergence to a cycle, earlier in this section.

The following result follows immediately from Theorems 5.2 and 5.6.

Corollary 5.3. *Let ξ be a T-periodic solution of a continuous-time autonomous system given by (5.20), with $T \in \mathbb{R}^+$. Consider the first return map R of the Poincaré section $\mathcal{P}$ through $\xi(0)$. Suppose that all eigenvalues λ_i of the Jacobian matrix of R at $\xi(0)$ satisfy $|\lambda_i| < 1$ for all $1 \leq i \leq n-1$. Then the periodic solution ξ is stable.*

The problem with this criterion for the stability of a periodic solution is that the return map is defined only implicitly and therefore it is not easy to calculate the eigenvalues of its Jacobian matrix directly. The following lemma and theorem provide an indirect method.

Lemma 5.1. *The linearization of the first return map R is $[P \cdot M(T)]_{\mathcal{P}}$ where $M(t)$ is the solution of the variational equations around the periodic solution $\xi(t)$, and where P is the orthogonal projection onto the Poincaré section $\mathcal{P}$, taking $\xi(0)$ as the origin.*

Proof:

Define the unit vector

$$e = \frac{\dot{\xi}(0)}{\|\dot{\xi}(0)\|} = \frac{F(\xi)(0)}{\|F(\xi)(0)\|}.$$

From linear algebra, we know that the orthogonal projection on $\mathcal{P}$ amounts to multiply a vector in $\mathbb{R}^n$ by the projection matrix

$$P = I_n - ee^T$$

with T denoting transposition. Let $\mathcal{U}$ be a sufficiently small neighborhood of $\xi(0)$ on $\mathcal{P}$, let $x_0 \in \mathcal{U}$ and let $x(t)$ be the solution with initial condition $x(0) = x_0$. Up to first order approximation, we have that

$$x(T) - \xi(T) \approx M(T)(x_0 - \xi(0)) \tag{5.36}$$

because of (5.35) where $x^\star = \xi$.

Because it starts from a sufficiently close neighborhood of $\xi(0)$ on $\mathcal{P}$, the solution x returns to $\mathcal{P}$ at a time $T + \Delta T$, slightly different from T. Hence

$$e^T \cdot (x(T + \Delta T) - \xi(T)) = 0 \tag{5.37}$$

because $x(T + \Delta T) - \xi(T) \in \mathcal{P}$ and e is orthogonal to $\mathcal{P}$. Now,

$$x(T + \Delta T) \approx x(T) + \dot{x}(T)\Delta T = x(T) + F(x(T))\Delta T \approx x(T) + F(\xi(T))\Delta T$$

and therefore

$$\begin{aligned} x(T + \Delta T) - \xi(T) &\approx x(T) - \xi(T) + F((x(T))\Delta T \\ &\approx M(T)(x_0 - \xi(0)) + F(\xi(T))\Delta T \end{aligned} \tag{5.38}$$

because of (5.36). Combining (5.37) and (5.38), and observing that

$$e^T F(\xi(T)) = e^T F(\xi(0)) = \frac{F^T(\xi)(0)F(\xi(0))}{\|F(\xi)(0)\|} = \|F(\xi)(0)\|$$

yields that

$$\Delta T \approx -\frac{e^T M(T)(x_0 - \xi(0))}{\|F(\xi)(0)\|}.$$

Substituting this back into (5.38) leads to

$$\begin{aligned} x(T + \Delta T) - \xi(0) &= x(T + \Delta T) - \xi(T) \\ &\approx M(T)(x_0 - \xi(0)) - \frac{F(\xi(T))e^T M(T)(x_0 - \xi(0))}{\|F(\xi)(0)\|} \\ &= M(T)(x_0 - \xi(0)) - ee^T \cdot M(T)(x_0 - \xi(0)) \\ &= \left(I_n - ee^T\right) M(T)(x_0 - \xi(0)) \\ &= PM(T)(x_0 - \xi(0)). \end{aligned}$$

■

The eigenvalues of $M(T)$ are called the *Floquet multipliers*. They are related to the eigenvalues of the return map by the following theorem.

Lemma 5.2. *Let ξ be a T-periodic solution of a continuous-time autonomous system given by (5.20), with $T \in \mathbb{R}^+$. Consider the first return map R of the Poincaré section $\mathcal{P}$ through $\xi(0)$ and the solution $M(t)$ of the variational equations (5.33) and (5.34) along $\xi(t)$. The eigenvalues of the Jacobian matrix of R at $\xi(0)$ are $\lambda_1, \ldots, \lambda_{n-1}$ if and only if the Floquet multipliers, i.e. the eigenvalues of $M(T)$, are $\lambda_1, \ldots, \lambda_{n-1}, 1$.*

Proof:

We extend the linearization of R at $\xi(0)$ to the whole space $\mathbb{R}^n$ by replacing $[PM(T)]_{\mathcal{P}}$ by $PM(T)P$. On $\mathcal{P}$, the two linear maps are identical, but $PM(T)P$ has the additional eigenvalue 0 with eigenvector e. Now, let v be an eigenvector of $M(T)$ with eigenvalue λ. Then, since for a projection matrix $P^2 = P$,

$$[PM(T)P]Pv = PM(T)Pv = PM(T)v - PM(T)(I_n - P)v.$$

Because $I_n - P$ projects onto the subspace spanned by e, and because e is an eigenvector with eigenvalue 1 of $M(T)$,

$$[PM(T)P]Pv = PM(T)v - P(I_n - P)v = PM(T)v = P(\lambda v) = \lambda Pv.$$

Therefore, as long as $Pv \neq 0$, λ is is also an eigenvalue of $PM(T)P$. But $Pv = 0$ implies that v is proportional to e. Consequently, $[PM(T)]_{\mathcal{P}}$ and $M(T)$ have the same eigenvalues, except that $M(T)$ has the additional eigenvalue 1 corresponding to the eigenvector e.

■

The solution $M(T)$ of the variational equation, which is a n-dimensional time-varying continuous-time system, has therefore $(n-1)$ eigenvalues in common with the return map R, which is a $(n-1)$-dimensional discrete-time system on $\mathcal{P}$. The eigenvector of the additional nth eigenvalue 1 of $M(T)$ is $\dot{\xi}(0)$, and is orthogonal to the Poincaré section $\mathcal{P}$. Indeed, let us consider two T-periodic solutions $\xi(t)$ and $x_\tau(t) = \xi(t+\tau)$. If $\tau > 0$ is sufficiently small, the difference between the two solutions can be approximated to the first order by

$$\Delta x(t) = x_\tau(t) - \xi(t) = \xi(t+\tau) - \xi(t) \approx \tau\dot{\xi}(t)$$

Now, because of (5.28),

$$\Delta x(t) = M(t)\Delta x(0) \approx M(t)\tau\dot{\xi}(0)$$

and therefore

$$\dot{\xi}(t) \approx M(t)\dot{\xi}(0).$$

In particular, taking $t = T$ in the previous equation, we find that $\dot{\xi}(T) \approx M(T)\dot{\xi}(0)$. Since ξ is a T-periodic solution, $\dot{\xi}(T) = \dot{\xi}(0)$, hence

$$\dot{\xi}(0) \approx M(T)\dot{\xi}(0),$$

which shows that $\dot{\xi}(0)$ is indeed an eigenvector of $M(T)$ with eigenvalue 1. Figure 5.13 shows a T-periodic solution $\xi(t)$ of the Van der Pol oscillator, together with the vectors $\dot{\xi}(t)$ for a number of time instants t. The solution matrix $M(t)$ of the variational equation (5.32) transforms the vector $\dot{\xi}(0)$ at time 0 in the vector $\dot{\xi}(t)$ at time t.

Lemma 5.2 shows that the stability of a periodic solution can be assessed from the computation of the eigenvalues of $M(t)$, as summarized by the first part of the following theorem (the last property, namely that the periodic solution is asymptotically stable except for a time shift, is not proven here).

Theorem 5.7. *Let ξ be a T-periodic solution of a continuous-time autonomous system given by (5.20), with $T \in \mathbb{R}^+$, and let Γ be its orbit. Consider the first return map R of the Poincaré section $\mathcal{P}$ through $\xi(0)$ and the solution $M(t)$ of the variational equations (5.33) and (5.34) along $\xi(t)$. If all the Floquet multipliers λ_i, $1 \leq i \leq n-1$ (the eigenvalues of $M(T)$), except one (which is $\lambda_n = 1$) satisfy $|\lambda_i| < 1$, the periodic solution $\xi(t)$ is stable. In addition, any solution starting sufficiently close to $\xi(0)$ converges to a time-shifted version of the periodic solution $\xi(t)$, i.e. its ω-limit set is Γ.*

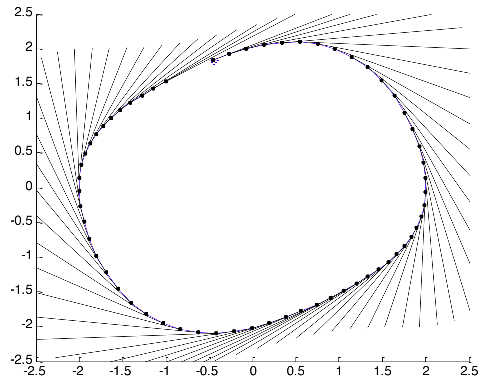

Figure 5.13: Periodic solution $\xi(t)$ of the Van der Pol oscillator with the vectors $\dot{\xi}(t)$ represented as originating from $\xi(t)$ for a number of instants t.

Example: Van der Pol Oscillator

Numerical solutions of the variational equations (5.32) of the Van der Pol oscillator with $\lambda = 0.3$ along the periodic solution yields the Floquet multipliers 1 and 0.15. Thus, solutions starting sufficiently close to the periodic orbit have this cyle as ω-limit set as well. This is illustrated in Figure 5.14. Actually, all solutions but the unstable equilibrium point at the origin have this property.

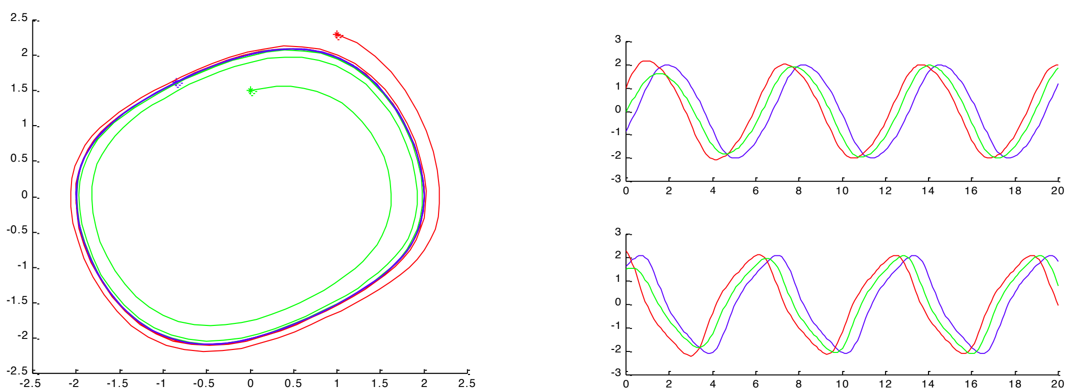

Figure 5.14: Three solutions of the Van der Pol oscillator, with $\lambda = 0.3$: Orbits (left) and temporal trajectories (right). The solution in blue is a periodic solution of the system, and clearly the red and green solutions converge to a time-shifted version of this periodic solution.

5.3.9 Unique Asymptotic Behavior: Contraction Approach

As we saw in the previous chapter, nonlinear dynamical systems may have much complex attractor than equilibrium/fixed points or periodic solutions. Unfortunately, the complexity is also present in the stability analysis. We present here a technique to establish the unique asymptotic behavior of solutions. We have to prove that the difference between any two solutions converges to zero. For this purpose, we represent the time evolution of this difference in a special way, which can be generalized to non-autonomous systems by using state- and time-dependent norms as in W.Lohmiller, J.-J.Slotine, On contraction analysis for nonlinear systems, Automatica, vol. 34, no.6, 1998.

Let x and $x^\star$ be two solutions of an autonomous dynamical system, and let us consider the following linear interpolation between these two solutions:

$$z(t,\mu) = (1-\mu)x(t) + \mu x^\star(t). \tag{5.39}$$

The mean value theorem for vector function states that for any continuously differentiable function $F : \mathbb{R}^n \to \mathbb{R}^n$, and any vector $y, h \in \mathbb{R}^n$,

$$F(y+h) - F(y) = \left(\int_0^1 J\left(y + \mu h\right) d\mu\right) \cdot h$$

where J is the Jacobian marix of F. Setting $y = x(t) = z(t,0)$ and $h = x^\star(t) - x(t) = z(t,1) - z(t,0)$, the previous relation becomes

$$F\left(x^\star(t)\right) - F\left(x(t)\right) = \left(\int_0^1 J\left(z(t,\mu)\right) d\mu\right) \cdot \left(x^\star(t) - x(t)\right). \tag{5.40}$$

Unique Asymptotic Behavior in Discrete-time Systems

Let first consider the case where x and $x^\star$ are two solutions of the autonomous discrete-time system (5.15). Then (5.40) yields that

$$\begin{aligned} x^\star(t+1) - x(t+1) &= F\left(x^\star(t)\right) - F\left(x(t)\right) \\ &= \left(\int_0^1 J\left(z(t,\mu)\right) d\mu\right) \cdot \left(x^\star(t) - x(t)\right), \end{aligned}$$

from which we obtain the following theorem.

Theorem 5.8. *Suppose that the Jacobian matrices $J(x(t))$ of the discrete-time system (5.15) are uniformly bounded, i.e. that there is some constant $K < 1$ such that for all $x \in \mathbb{R}^n$*

$$\|J(x)\| \le K < 1$$

where $\|\cdot\|$ is any matrix norm induced by a vector norm. Then the system has unique asymptotic behavior. More precisely, for any two solutions x and $x^\star$ of (5.15), and for all $t \in \mathbb{N}$,

$$\|x^\star(t) - x(t)\| \le K^t \left\|x^\star(0) - x(0)\right\|. \tag{5.41}$$

Proof:

$$\begin{aligned} \|x^\star(t+1) - x(t+1)\| &= \left\|\int_0^1 J\left(z(t,\mu)\right)\right\| \cdot \|x^\star(t) - x(t)\| \\ &\le \left(\int_0^1 \left\|J\left(z(t,\mu)\right)\right\|\right) \cdot \|x^\star(t) - x(t)\| \\ &\le \left(\int_0^1 K d\mu\right) \cdot \|x^\star(t) - x(t)\| = K\left\|x^\star(t) - x(t)\right\|. \end{aligned}$$

Iterating this expression for all times from t to 0 yields (5.41).

■

Unique Asymptotic Behavior in Continuous-time Systems

Now x and $x^\star$ are two solutions of the autonomous continuous-time system (5.20). Then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|x^\star(t)-x(t)\right\|^2 &= \frac{1}{2}\frac{d}{dt}\left(x^\star(t)-x(t)\right)^T\left(x^\star(t)-x(t)\right)\\
&= \frac{1}{2}\left[\left(\dot{x}^\star(t)-\dot{x}(t)\right)^T\left(x^\star(t)-x(t)\right)+\left(x^\star(t)-x(t)\right)^T\left(\dot{x}^\star(t)-\dot{x}(t)\right)\right]\\
&= \frac{1}{2}\left[\left(F(x^\star(t))-F(x(t))\right)^T\left(x^\star(t)-x(t)\right)+\left(x^\star(t)-x(t)\right)^T\left(F\left(x^\star(t)\right)-F(x(t))\right)\right].
\end{aligned}$$

Because of (5.40), we find next that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|x^\star(t)-x(t)\right\|^2 &= \frac{1}{2}\left[(x^\star(t)-x(t))^T\left(\int_0^1 J(z(t,\mu))d\mu\right)^T(x^\star(t)-x(t)) + \right.\\
&\qquad \left.(x^\star(t)-x(t))^T\left(\int_0^1 J(z(t,\mu))d\mu\right)(x^\star(t)-x(t))\right]\\
&\quad \frac{1}{2}\left[(x^\star(t)-x(t))^T\left(\int_0^1\left(J^T(z(t,\mu))+J(z(t,\mu))\right)d\mu\right)(x^\star(t)-x(t))\right].
\end{aligned}$$

The following theorem follows from this representation. Note that the symmetric part of a matrix A is $(A+A^T)/2$.

Theorem 5.9. *Suppose that symmetric part of the Jacobian matrices $J(x(t))$ of the continuous-time system (5.20) are uniformly negative definite, i.e. that there is some constant $\kappa<1$ such that for all $x\in\mathbb{R}^n$*

$$\left(J(x)+J^T(x)\right)/2\leq\kappa<0.$$

Then the system has unique asymptotic behavior. More precisely, for any two solutions x and $x^\star$ of (5.20), and for all $t\in\mathbb{R}^T$,

$$\left\|x^\star(t)-x(t)\right\|\leq e^{-\kappa t}\left\|x^\star(0)-x(0)\right\|. \tag{5.42}$$

where $\|\cdot\|$ is the Euclidean norm.

Proof:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|x^\star(t)-x(t)\right\|^2 &= \frac{1}{2}\left[(x^\star(t)-x(t))^T\left(\int_0^1\left(J\left(z(t,\mu)\right)d\mu\right)^T+\left(\int_0^1 J\left(z(t,\mu)\right)\right)d\mu\right)(x^\star(t)-x(t))\right]\\
&\leq (x^\star(t)-x(t))^T\left(\int_0^1\kappa d\mu\right)(x^\star(t)-x(t))\\
&= \kappa\left\|x^\star(t)-x(t)\right\|^2,
\end{aligned}$$

which yields directly (5.42).

■