3

Observability and Controllability in Linear Systems

Chapter 2 focused on the stability of a linear system whose state equations are given by (2.1). In this chapter, we focus more on the input-output relation of the system, and will answer a couple of questions linked to the observability of the internal state of the system from the input and output of the system, and to the possibility to drive the system to a prescribed state by applying a chosen input.

3.1 Observability

3.1.1 Introduction

If we know the state equations (2.2) (or (2.3)) of the system and the input signal u at time t, and if we observe the output signal y at time t, can we determine the state x at time t? Remember that the output $y(t)$ is obtained from (2.2), which we recall here:

$$y(t) = Cx(t) + Du(t).$$

If C is invertible, then the answer to this question is positive. However, in general, C is not invertible. Actually, in the case we consider in this chapter, the output signal is univariate (has scalar values; the output space is $\Theta = \mathbb{R}$) and the states are of arbitrary dimension n, the matrix C is $1 \times n$ and for any $n > 1$ is certainly not invertible. Therefore, we have to modify the question.

If we know the state equations (2.2) (or (2.3) of the system and the input signal u during the time interval $[t, t+T]$, and if we observe the output signal y during the same time interval $[t, t+T]$, can we determine the state x at time t?

Since we consider only time-invariant systems, we can set $t = 0$ without loss of generality. Let us start by answering the question with a particular example, before moving to the general answer.

3.1.2 Example: Mass-spring system in Newtonian mechanics

Let us return to system introduced in Section 1.3.1, which is a mass m attached to a wall by a spring of constant k, and to which an external force $f(t)$ is applied to pull the mass away from the wall, as shown in Figure 1.1. The input signal is the external force $f(t)$, the state equations are given by (1.1) and (1.2), which are recalled here

$$\frac{dx_1}{dt}(t) = x_2(t) \tag{3.1}$$

$$\frac{dx_2}{dt}(t) = -\frac{k}{m}x_1(t) + \frac{1}{m}f(t) \tag{3.2}$$

$$y(t) = x_1(t). \tag{3.3}$$

The solution starting from arbitrary solutions is given by (1.16) and (1.17), which read

$$\begin{aligned} y(t) = x_1(t) &= \left(x_1(0) - \frac{f}{k}\right)\cos(\omega t) + \frac{x_2(0)}{\omega}\sin(\omega t) + \frac{f}{k} \\ x_2(t) &= -\left(x_1(0) - \frac{f}{k}\right)\omega\sin(\omega t) + x_2(0)\cos(\omega t) \end{aligned}$$

with $\omega = \sqrt{k/m}$. It follows that

$$\begin{aligned} x_1(0) &= y(0) \\ x_2(0) &= -\omega\left(y\left(\frac{\pi}{2\omega}\right) - \frac{f}{k}\right). \end{aligned}$$

Therefore, if we choose $T = \pi/2\omega$, we can find the initial state. The answer to the question is positive.

We can actually decrease T, because, we can observe that

$$\begin{aligned} x_1(0) &= y(0) \\ x_2(0) &= \dot{y}(0) \end{aligned}$$

and in order to determine $\dot{y}(0)$, it is sufficient to know $y(t)$ for $t \in [0, \varepsilon]$ for any $\varepsilon > 0$ because

$$\dot{y}(0) = \lim_{\delta \to 0} \frac{y(\delta) - y(0)}{\delta}.$$

3.1.3 Definition

The previous discussion leads to the following definition of observable system, as a system where the initial state can be inferred from knowing the output $y(t)$ (and input $u(t)$ is this one is nonzero) which is the following.

Definition 3.1 (Observability)**.** *A system is observable if there exists a finite time $T > 0$ such that the free response signal in the time interval $[0, T]$ determines the initial state $x(0)$.*

Knowing $x(0)$ (and the input $u(t)$) we can then recover the state at all times $t \geq 0$.

3.1.4 Criterion

We are now going to state a necessary and sufficient condition for a linear system to be observable, which will be expressed in terms of the *observability matrix* of the linear system, constructed from the matrices A and C of the "ABCD" representation of a linear system:

$$M_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}. \tag{3.4}$$

Theorem 3.1. *A linear system is observable if and only if the rank of its observability matrix M_o is n.*

The theorem holds both for linear continuous-time systems and discrete-time systems. For continuous-time systems, the initial state can be determined from the free response in an arbitrarily small time interval $t \in [0, \varepsilon]$, for any $\varepsilon > 0$. For discrete-time systems, the initial state can be determined from the first n samples of the output signal of the free response. We prove the theorem for continuous-time systems when the dimension of the output space is 1; the proof for discrete-time systems is similar (with the derivative $d^k y/dt^k(0)$ replaced by $y(t-k)$).

Proof:

($\Leftarrow$) Because of (2.10) and of (2.11), the free response $y(t) = Ce^{At}x(0)$ and therefore

$$\begin{aligned} y(0) &= Cx(0) \\ \frac{dy}{dt}(0) &= CAx(0) \\ \frac{d^2y}{dt^2}(0) &= CA^2x(0) \\ &\vdots \\ \frac{d^{n-1}y}{dt^{n-1}}(0) &= CA^{n-1}x(0) \end{aligned}$$

whence

$$x(0) = M_0^{-1} \cdot \begin{bmatrix} y(0) \\ \frac{dy}{dt}y(0) \\ \frac{d^2y}{dt^2}(0) \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}}(0) \end{bmatrix}.$$

It is sufficient to know the output signal $y(t)$ for $t \in [0, \varepsilon]$ for an arbitrarily small $\varepsilon > 0$. This proves the first part of the theorem.

($\Rightarrow$) We prove the necessity of the condition by contradiction. Suppose that M_o is non invertible. Then there is a vector $v \neq 0$ such that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \cdot v = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.5}$$

Let us pick the initial condition $x(0)$ as this vector v. If $C = 0$ then $y(t) = 0$ for all time $t \in \mathbb{R}^+$ and clearly, $x(0) = v$ cannot be recovered from y. Let $C \neq 0$ and let $\mathcal{V}$ be the subspace of all vectors in $\mathbb{R}^n$ that are orthogonal to the row vectors of C; the dimension of $\mathcal{V}$ is at most $n-1$ because $C \neq 0$. Now, (3.5) yields that $x(0), Ax(0), \ldots, A^{n-1}x(0) \in \mathcal{V}$. These n vectors cannot be linearly independent, because the dimension of of $\mathcal{V}$ is at most $n-1$.

(i) We can then apply the Cayley-Hamilton theorem to prove that A^k with $k \geq n$ can always be written as a linear combination of $I_n, A, A^2, \ldots, A^{n-1}$ and therefore that $A^k x(0)$ is a linear combination of $x(0), Ax(0), \ldots, A^{n-1}x(0)$. Therefore $A^k x(0) \in \mathcal{V}$ for all $k \geq n$.

(ii) An alternative way to recover the same result if we do not want to invoke the Cayley-Hamilton theorem, is to find that since $x(0), Ax(0), \ldots, A^{n-1}x(0)$ are not linearly independent, there are real constants $\mu_0, \ldots, \mu_{n-1}$ such that

$$\sum_{i=0}^{n-1} \mu_i A^i x(0) = 0.$$

Let $m \leq n-1$ be such that $\mu_m \neq 0$ and $\mu_i = 0$ for $m+1 \leq i \leq n-1$. Then

$$A^m x(0) = -\sum_{i=0}^{m-1} \frac{\mu_i}{\mu_m} A^i x(0). \tag{3.6}$$

Multiplying this equation by A, and using it later to replace $A^m x(0)$, we get

$$\begin{aligned}
A^{m+1}x(0) &= -\sum_{i=0}^{m-1} \frac{\mu_i}{\mu_m} A^{i+1} x(0) \\
&= -\sum_{j=1}^{m} \frac{\mu_{j-1}}{\mu_m} A^j x(0) \\
&= -\frac{\mu_{m-1}}{\mu_m} A^m x(0) - \sum_{j=1}^{m-1} \frac{\mu_{j-1}}{\mu_m} A^j x(0) \\
&= \frac{\mu_{m-1}}{\mu_m} \sum_{j=0}^{m-1} \frac{\mu_j}{\mu_m} A^j x(0) - \sum_{j=1}^{m-1} \frac{\mu_{j-1}}{\mu_m} A^j x(0) \\
&= \sum_{j=0}^{m-1} \frac{\mu_{m-1}\mu_j - \mu_m \mu_{j-1}}{\mu_m^2} A^j x(0)
\end{aligned}$$

where we have set $\mu_{-1} = 0$. This shows that $A^{m+1}x(0)$ is a linear combination of $A^i x(0)$ for $0 \leq i \leq m-1$ and is therefore belonging to $\mathcal{V}$. Repeating this argument, we find that $A^k x(0)$ is a linear combination of $A^i x(0)$ for $0 \leq i \leq m-1$ and is therefore belonging to $\mathcal{V}$ for all $k \in \mathbb{N}$.

Using either (i) or (ii), we have therefore that $A^k x(0) \in \mathcal{V}$ for all $k \in \mathbb{N}$, which yields that

$$\begin{aligned} \exp(At)x(0) &= \left(I_n + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots \right) x(0) \\ &= x(0) + t(Ax(0)) + \frac{t^2}{2!}(A^2x(0)) + \frac{t^3}{3!}(A^3x(0)) + \ldots \end{aligned}$$

and therefore that $\exp(At)x(0) \in \mathcal{V}$ as well. This means that the output signal y is identically zero when the initial state $x(0)$ is chosen as a vector v verifying (3.5). As a result, the initial state $x(0)$ cannot be found from y if M_o is not invertible. This contradicts the assumption that the system is observable. Consequently, M_o must be invertible.

■

An alternative result that can be used to verify that a system is observable is the PBH (Popov-Belevitch-Hautus) test, which is as follows. Remember that the rank of a matrix is the maximum number of linearly independent rows, or the maximum number of linearly independent columns, of this matrix. We state and prove the theorem when the eigenvalues of A are real; it is also valid when the eigenvalues of A are complex but one needs then to take $s \in \mathbb{C}$ and adapt the proof of the theorem accordingly.

Theorem 3.2. *Suppose that all the eigenvalues of A are real and that at least one is nonzero. A linear system is observable if and only if for all $s \in \mathbb{R}$, the matrix*

$$N_o = \begin{bmatrix} sI_n - A \\ C \end{bmatrix}. \tag{3.7}$$

has full rank (i.e. its rank is equal to n).

Proof:

($\Rightarrow$) Suppose that the rank of N_o is less than n for some $s \in \mathbb{R}$. Then there is a nonzero vector $v \in \mathbb{R}^n$ such that

$$\begin{bmatrix} sI_n - A \\ C \end{bmatrix} \cdot v = \begin{bmatrix} sv - Av \\ Cv \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

whence $Av = sv$ and $Cv = 0$. Therefore

$$\begin{aligned} Cv &= 0 \\ CAv &= C(sv) = sCv = 0 \\ CA^2v &= CA(Av) = CA(sv) = sCAv = 0 \\ &\vdots \\ CA^{n-1}v &= 0, \end{aligned}$$

which shows that there is a nonzero vector v such that $M_o v = 0$, implying that M_o is non invertible. This is impossible because the system is observable, and because of Theorem 3.1.

($\Leftarrow$) Suppose that the system is non observable, and let v be an eigenvector of A, with the corresponding eigenvalue $\lambda \neq 0$. Then (i) $(sI_n - A)v = 0$ for $s = \lambda$, and (ii) $\lambda Cv = C(\lambda v) = CAv = 0$, which yields that $Cv = 0$. Therefore $N_o v = 0$, which shows that the rank of N_0 is less than n.

■

In general, A is a regular matrix (i.e. its rank is n), and it suffices therefore to verify the rank of N_o when s is an eigenvalue of A. When s is not an eigenvalue of A, the rank of $sI_n - A$ is always n and so is then the rank of N_o because of (3.7).

Finally, a last criterion is stated in terms of the observability Gramian, which is defined as

$$W_{ot} = \int_0^t \exp(A^T\tau)C^T C \exp(A\tau)d\tau \tag{3.8}$$

or continuous-time linear time-invariant systems and by

$$W_{ot} = \sum_{\tau=0}^{t-1} (A^T)^\tau C^T C A^\tau \tag{3.9}$$

for discrete-time linear time-invariant systems. The following criterion, which is equivalent to Theorem 3.1, can be stated in terms of the observability Gramian:

Theorem 3.3. *A linear system is observable if and only if its observability Gramian W_{ot} is invertible at all times t.*

3.1.5 Examples

Example 1: Mass-spring system in Newtonian mechanics

The matrices A and C of the system given by (3.1), (3.2) and (3.3) are

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

from which we compute

$$M_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is clearly invertible. This confirms the observability of the system.

Example 2: A BIBO stable system with an unstable free system

The matrices A and C of the system given by (2.37), (2.38) and (2.39) are

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

from which we compute

$$M_o = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix},$$

which is invertible. The system is observable.

Example 3: Free Frictionless Motion

Consider the free frictionless motion, whose state equations are

$$\begin{aligned} \frac{dx_1}{dt}(t) &= x_2(t) \\ \frac{dx_2}{dt}(t) &= 0 \end{aligned}$$

with the output $y(t) = x_1(t)$ denoting the position of the mass at time t. The matrices A and C of the system are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

from which we compute

$$M_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is clearly invertible. The system is observable. If instead the output is the speed, i.e. $y(t) = x_2(t)$, then $C = [0 \ \ 1]$ and

$$M_o = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which is not invertible. The system is not observable (and indeed, it impossible to recover the initial position $x_1(0)$ by only observing the speed).

3.2 Controllability

3.2.1 Definition

If we know the state equations (2.2) (or (2.3)) of the system, can we bring the state from a value $x(0)$ to another value $x(T)$ in a finite time T by applying a chosen input signal u during the time interval $[0, T]$? This question is addressed here. It can be viewed as the dual question to that of observability.

Definition 3.2 (Controllability)**.** *A system is (completely) controllable if there is a finite time T such that it can be brought from any initial state $x(0) = x_0$ to any state $x(T) = x_1$ by applying a suitable input signal $u(t)$ for $t \in [0, T]$.*

Formally speaking, this definition applies only to a *completely controllable* system. A system is often said to be (simply) *controllable* if it can be brought from any initial state $x(0) = x_0$ to the origin (i.e $x(T) = 0$) in a finite time T by applying a suitable input signal u during the time interval $[0, T]$. A system is *reachable* if it can be brought from the initial state $x(0) = 0$ to an arbitrary state $x(T) = x_1$ in a finite time T by applying a suitable input signal u during the time interval $[0, T]$. It can be easily shown that for continuous-time systems, as well as discrete-time systems with an invertible state matrix A, reachability is equivalent to controllability, and this implies that reachable and controllable system verify Defintion 3.2. We will speak of controllable systems from now on to designate completely controllable systems.

3.2.2 Criterion

We can now state a necessary and sufficient condition for a linear system to be controllable, which will be expressed in terms of the *controllability matrix* of the linear system, constructed from the matrices A and B of the "ABCD" representation of a linear system, as follows:

$$M_c = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}. \tag{3.10}$$

Theorem 3.4. *A linear system is controllable if and only if the rank of its controllability matrix M_c is n.*

The theorem holds both for linear continuous-time systems and discrete-time systems. We give a short proof for discrete-time systems when the dimension of the output space is 1; the proof for continuous-time systems is similar but slightly more complex.

Proof:

(⇐) Because of (2.7),

$$\begin{aligned} x(t) - A^t x(0) &= \sum_{\tau=0}^{t-1} A^{t-\tau-1} B u(\tau) \\ &= A^{t-1}Bu(0) + A^{t-2}Bu(1) + \ldots + ABu(t-2) + Bu(t-1) \\ &= \begin{bmatrix} B & AB & \ldots & A^{t-2}B & A^{t-1}B \end{bmatrix} \cdot \begin{bmatrix} u(t-1) \\ u(t-2) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} \end{aligned}$$

For $t = n$, $\begin{bmatrix} B & AB & \ldots & A^{t-2}B & A^{t-1}B \end{bmatrix}$ is the square matrix M_c. If it is invertible, the input that drives the system from $x(0) = x_0$ to $x(T) = x_1$ in $T = n$ time steps is given by

$$\begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} = M_c^{-1} \cdot (x_1 - A^n x_0)\,.$$

(⇒) : If M_c is not invertible, the columns of M_c are linearly dependent and the range of M_c (the subspace of $\mathbb{R}^n$ that is spanned by these columns) is not the whole space $\mathbb{R}^n$. State vectors that are not in the range can therefore not be obtained by any input signal within time n. To prove that it cannot be obtained even after time n, we prove in an analogous fashion as for observability (or use the Cayley Hamilton theorem), that all column vectors of the form A^tB for $t \in \mathbb{N}$ are linear combinations of the vectors A^jB for $0 \leq j \leq n-1$. Therefore, they all belong to the range of M_c and therefore any state not in this range cannot be reached by applying an input signal, even when waiting for a long time.

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Similarly to observability, an alternative result that can be used to verify that a system is controllable is the PBH (Popov-Belevitch-Hautus) test, which is as follows. We state the theorem when the eigenvalues of A are real; it is also valid when the eigenvalues of A are complex but one needs then to take $s \in \mathbb{C}$ and adapt the proof of the theorem accordingly.

Theorem 3.5. *Suppose that all the eigenvalues of A are real and that at least one is nonzero. A linear system is controllable if and only if for all $s \in \mathbb{R}$, the matrix*

$$N_c = \begin{bmatrix} sI_n - A & B \end{bmatrix}. \tag{3.11}$$

has full rank (i.e. its rank is equal to n).

Finally, a last criterion is stated in terms of the controllability Gramian, which is defined as

$$W_{ct} = \int_0^t \exp(A\tau)BB^T \exp(A^T\tau)d\tau \tag{3.12}$$

for continuous-time linear time-invariant systems and by

$$W_{ct} = \sum_{\tau=0}^{t-1} A^\tau BB^T (A^\tau)^T \tag{3.13}$$

for discrete-time linear time-invariant systems. The following criterion, which is equivalent to Theorem 3.4, can be stated in terms of the controllability Gramian:

Theorem 3.6. *A linear system is controlable if and only if its controllability Gramian W_{ct} is invertible at all times t.*

We note that no constraint is posed on the input signal u transferring the state x from x_0 at time 0 to a x_1 at time T. It makes sense to look for the minimum energy input that controls the system from x_0 at time 0 to x_1 at time t_1, i.e. the input $u^\star(\cdot)$ that minimizes, for a continuous-time system,

$$\int_0^{t_1} \|u(\tau)\|_2^2 d\tau \tag{3.14}$$

where $\|u(\tau)\|_2 = u^T(\tau) \cdot u(t)$, with a similar expression in discrete-time. This is called minimum energy control, and the solution is given by using the controllability Gramian, which we state and prove for continuous-time systems.

Theorem 3.7. *The minimal energy input that transfers a linear continuous-time controllable system from $x(0) = x_0$ at time 0 to $x(t_1) = x_1$ at time t_1 is*

$$u^\star(t) = -B^T \exp(A^T(t_1 - t))W_{ct_1}^{-1} (\exp(At_1)x_0 - x_1) \tag{3.15}$$

for $0 \leq t \leq t_1$, where W_{ct_1} is the controllability Gramian given by (3.12).

Proof:

The solution of the system at time t_1 is

$$x(t_1) = e^{At_1}x(0) + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau \tag{3.16}$$

By setting $x(0) = x_0$ and by plugging (3.15) in (3.16), we find that $x(t_1) = x_1$, hence the input $u^\star(\cdot)$ transfers x_0 at time 0 to x_1 at time t_1.

Now, let $u(\cdot)$ be another input that transfers x_0 at time 0 to x_1 at time t_1. Denote by $\overline{x} = e^{-At_1}x_1 - x_0$. Because $x(0) = x_0$ and $x(t_1) = x_1$ in (3.16), we can express $\overline{x}$ as

$$\overline{x} = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

and as

$$\overline{x} = \int_0^t e^{-A\tau}Bu^\star(\tau)d\tau,$$

so that substracting both equations yields

$$0 = \int_0^t e^{-A\tau}B\left(u(\tau) - u^\star(\tau)\right)d\tau. \tag{3.17}$$

Now, we have that

$$\begin{aligned}
\int_0^{t_1} \|u(\tau)\|^2 d\tau &= \int_0^{t_1} \|u(\tau) - u^\star(\tau) + u^\star(\tau)\|^2 d\tau \\
&= \int_0^{t_1} \|u(\tau) - u^\star(\tau)\|^2 d\tau + \int_0^{t_1} \|u^\star(\tau)\|^2 d\tau + 2\int_0^{t_1} \left(u(\tau) - u^\star(\tau)\right)^T u^\star(\tau)d\tau \\
&= \int_0^{t_1} \|u(\tau) - u^\star(\tau)\|^2 d\tau + \int_0^{t_1} \|u^\star(\tau)\|^2 d\tau
\end{aligned}$$

because of (3.17) yields that

$$\begin{aligned}
&\int_0^{t_1} (u(\tau) - u^\star(\tau))^T u^\star(\tau) d\tau \\
&= -\left(\int_0^{t_1} (u(\tau) - u^\star(\tau))^T B^T \exp(A^T(t_1 - \tau)) d\tau\right) W_{ct_1}^{-1} (\exp(At_1)x_0 - x_1) \\
&= -\exp(At_1) \left(\int_0^{t_1} \exp(-A\tau) B \left(u(\tau) - u^\star(\tau)\right) d\tau\right) W_{ct_1}^{-1} (\exp(At_1)x_0 - x_1) \\
&= 0.
\end{aligned}$$

Therefore, for any $u(\cdot)$ transferring x_0 at time 0 to x_1 at time t_1,

$$\int_0^{t_1} \|u(\tau)\|^2 d\tau \geq \int_0^{t_1} \|u^\star(\tau)\|^2 d\tau,$$

which proves the claim.

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3.2.3 Examples

Example 1: Mass-spring system in Newtonian mechanics

The matrices A and B of the system given by (3.1) and (3.2) are

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix},$$

from which we compute

$$M_c = \begin{bmatrix} 0 & 1/m \\ 1/m & 1/m \end{bmatrix},$$

which is invertible. The system is controllable.

Example 2: A BIBO stable system with an unstable free system

The matrices A and B of the system given by (2.37) and (2.38) are

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

from which we compute

$$M_c = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix},$$

which is not invertible. The system is not controllable. Indeed, remember that the system is not only observable but also BIBO stable, hence any input signal will result in a bounded response. However we saw that the free stable is unstable. One can expect therefore that there are non zero initial states that will lead to solutions diverging to infinity, and that no input can bring these solutions to zero.

3.3 Connection with Transfer Functions

The transfer function of the last example with the system given by (2.37) and (2.38), which we just shown to be uncontrollable, was computed in the previous chapter, and reads

$$H(s) = \frac{1}{s+1}.$$

It has one pole in $s = -1$, hence it is BIBO stable. However, we saw that the natural frequencies of the systems, given by the eigenvalues of A, are $\lambda_1 = -1$ and $\lambda_2 = 1$. If we explicit the computation of the transfer function from (2.34), we find that

$$\begin{aligned} H(s) &= C(sI_n - A)^{-1}B + D \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} s-1 & 0 \\ -1 & s+1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{s-1}{(s-1)(s+1)} \\ &= \frac{1}{s+1}. \end{aligned}$$

The pole in -1 also appears as a zero of the numerator of $H(s)$, and is therefore cancelled: one speaks of a pole-zero cancellation. As a result, the degree of the denominator of $H(s)$ is smaller than the order $n = 2$ of the system. This is always the symptom that the system is unobservable and/or uncontrollable, in the sense of the following theorem, which we do not prove here.

Theorem 3.8. *Consider a continuous-time linear system with an input-output transfer function $H(s)$. If $H(s)$ has a pole-zero cancellation, then the system is unobservable and/or uncontrollable. Conversely, if $H(s)$ does not have a pole-zero cancellation, then the system is observable and controllable.*

The same result holds for a discrete-time system with transfer function $H(z)$ given by (2.35).