6

Periodic Solutions in Planar Systems

6.1 Limit Cycle

It is in general easy to find equilibrium points of a continuous-time system, it is in general difficult to find periodic solutions, even by simulation. For 2-dim systems, there are however two useful theorems that can be used to predict the presence or the absence of periodic solutions. In this chapter, we consider therefore only systems of the form

$$\dot{x}_1 = F_1(x_1, x_2) \quad (6.1)$$

$$\dot{x}_2 = F_2(x_1, x_2). \quad (6.2)$$

We are looking for non-trivial periodic solutions of such systems, i.e. for T-periodic solutions with $T > 0$. The orbit of such solution, which we denote by Γ, is called a *cycle*. It is called a *limit cycle* if it is the α-limit set or ω-limit set of some solution whose initial condition is not on the cycle. Asymptotically stable and unstable periodic orbits are examples of limit cycles. Conversely, the orbits of a center of a linear system are cycles, but not limit cycles.

Let us first recall the following result, which establishes the link between periodic solutions and closes orbits of a continuous-time 2-dim system.

Theorem 6.1 (Closed Orbit)**.** *Let Γ be the orbit of the solution $\xi(t)$ of (6.1)-(6.2) starting from $\xi(0) = \xi_0$. Then $\xi(t)$ is a non-trivial periodic solution of (6.1)-(6.2) if and only if Γ is a closed curve containing no equilibrium point.*

Proof:

($\Rightarrow$) If $\xi(t)$ is a T-periodic solution, with $T > 0$, then $\xi(t+T) = \xi(t)$ for all $t \in \mathbb{R}$ and in particular $\xi(T) = \xi(0) = \xi_0$, making the orbit of $\xi(t)$ the closed curve Γ. Its omega-limit set is also Γ. If Γ was containing an equilibrium point, the omega-limit set of ξ would that equilibrium point, and not the closed curve Γ. Therefore Γ is a closed curve containing no equilibrium point.

($\Leftarrow$) Suppose that the orbit Γ is a closed curve containing no equilibrium point. Let us pick a point $\xi_0 \in \Gamma$, and let us consider the solution $\xi(t)$ of (6.1)-(6.2) with initial condition $\xi(0) = \xi_0$. Then $\xi(t)$ moves along Γ with velocity $\dot{\xi}(t) \neq 0$ since there is no equilibrium on Γ. As $F(x) = (F_1(x_1, x_2), F_2(x_1, x_2))$, the norm of the velocity is

$$\left\|\dot{\xi}(t)\right\|_2 = \|F(\xi(t))\|_2 = \sqrt{F_1^2(\xi_1(t), \xi_2(t))) + F_2^2(\xi_1(t), \xi_2(t)))}. \tag{6.3}$$

Let $F_{\min}$ denote the minimum of the right hand side of (6.3) over Γ. Note that $F_{\min} > 0$ because $\dot{\xi}(t) \neq 0$. Therefore, there is some $0 < T < 1/F_{\min}$ such that $\xi(T) = \xi_0 = \xi(0)$, which yields that ξ is T-periodic.

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6.2 Absence of a periodic solution

A *connected* domain $\mathcal{D}_c$ of $\mathbb{R}^2$ is a set in which every two points in $\mathcal{D}_c$ can be connected by a curve lying entirely within $\mathcal{D}_c$. A set $\mathcal{D} \subseteq \mathbb{R}^2$ is *simply connected* if it is connected, and if any curve between two points can be continuously contracted, staying within $\mathcal{D}$, into another curve with the same endpoints. More intuitively, a set $\mathcal{D} \subseteq \mathbb{R}^2$ is simply connected if it is connected and has no holes in it. For instance, an annular region is connected but not simply connected. An ellipse, a rectangle, etc are simply connected.

The *divergence* of function $F(x) = (F_1(x_1, x_2), F_2(x_1, x_2))$ is defined as the quantity

$$\text{div}F(x) = \frac{\partial F_1}{\partial x_1}(x_1, x_2) + \frac{\partial F_2}{\partial x_2}(x_1, x_2). \tag{6.4}$$

Theorem 6.2 (Bendixson's Theorem)**.** *Let $\mathcal{D}$ be a simply connected of $\mathbb{R}^2$ such that the divergence of F, given by (6.4), is not identically zero over any subregion of $\mathcal{D}$, and does not change sign in $\mathcal{D}$. Then $\mathcal{D}$ does not contain any cycle of (6.1)-(6.2).*

Proof:

Suppose that J is a closed orbit of (6.1)-(6.2). Then for each point $x = (x_1, x_2) \in J$, $F(x)$ is tangent to J. Let $n(x)$ denote the outward directed unit vector that is orthogonal to j at x (i.e orthogonal to $F(x)$). Then $n^T(x)F(x) = 0$ for all $x \in J$, and thus

$$\int_J n^T(x)F(x)dx = 0.$$

The divergence theorem of Gauss allows us to write

$$\int_J n^T(x)F(x)dx = \iint_{\mathcal{S}} \text{div}F(x_1, x_2)dx_1dx_2$$

where $\mathcal{S}$ is the area enclosed by J. For the two members of this equation to be zero, one would need either to have that $\text{div}F(x) = 0$ for all $x \in \mathcal{S}$, or that $\text{div}F(x)$ changes sign over the region $\mathcal{S}$. But since $\mathcal{S} \subseteq \mathcal{D}$, neither can happen. Therefore $\mathcal{D}$ does not contain any closed orbit of (6.1)-(6.2).

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Example 1

Consider the system (from S. Sastry, p. 43)

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1 x_2^2 \\ \dot{x}_2 &= -x_1 + x_1^2 x_2.\end{aligned}$$

The linearization of this system around $(0,0)$ is inconclusive, because the eigenvalues of the Jacobian matrix are $\pm j$. However,

$$\text{div}F(x) = x_1^2 + x_2^2$$

and we see that $\text{div}F(x) > 0$ whenever $(x_1, x_2) \neq (0,0)$. Therefore this system has no periodic solution (other than the trivial solution $(x_1, x_2) = (0,0)$).

Example 2

Consider now the system

$$\begin{aligned}\dot{x_1} &= -x_2 + \alpha x_1 (x_1^2 + x_2^2 - 1) \\ \dot{x_2} &= x_1 + \alpha x_2 (x_1^2 + x_2^2 - 1)\end{aligned}$$

with $\alpha \neq 0$. One can compute that

$$\text{div}F(x) = 2\alpha \left(2\left(x_1^2 + x_2^2\right) - 1\right)$$

Let $\mathcal{D}_1$ be the disk

$$\mathcal{D}_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < \frac{1}{2}\right\}.$$

Observe that $\text{div}F(x)$ is not identically zero on any subregion of $\mathcal{D}_1$ (but at the equilibrium $(0,0)$), and is of the same sign on $\mathcal{D}_1$. Since $\mathcal{D}_1$ is simply connected, this system has no periodic solution in $\mathcal{D}_1$ (other than the trivial solution $(x_1, x_2) = (0,0)$).

Now, let $\mathcal{D}_2$ be the annular region

$$\mathcal{D}_2 = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid \frac{1}{2} < x_1^2 + x_2^2 < 2\right\}.$$

$\text{div}F(x)$ is again everywhere non zero and of the same sign on $\mathcal{D}_2$. However, $\mathcal{D}_2$ is connected, but not simply connected, and we cannot apply Bendixson's theorem for this region.

6.3 Existence of a periodic solution

The following theorem is given without proof. Remember that a compact set is a set that is both closed and bounded.

Theorem 6.3 (Poincaré-Bendixson's Theorem)**.** *Let $x(t)$ be a solution of (6.1)-(6.2), and let $\mathcal{S}_\omega$ denote its ω-limit set. If $\mathcal{S}_\omega$ is contained in a compact region $\mathcal{M} \subset \mathbb{R}^2$, and if $\mathcal{M}$ does not contain any equilibrium point of the system, then $\mathcal{S}_\omega$ is a cycle of (6.1)-(6.2).*

The difficulty of the application of the theorem is to find a region $\mathcal{M}$ that is forward invariant (and thus contains the ω-limit set of the solution), without containing any equilibrium point. Often, the best approach is to find a compact region $\mathcal{M}$, without any equilibrium, such that all solutions enter in it and do not leave it. In practice, $\mathcal{M}$ is often an annular region surrounding an equilibrium.

The theorem allows us to conclude that every compact and forward-invariant set $\mathcal{M}$ contains either an equilibrium point or a cycle. We cannot say however that the only ω-limit sets of points in $\mathcal{M}$ are equilibrium points or cycles. Indeed, the ω-limit sets of points in $\mathcal{M}$ may also be a union of orbits connecting many equilibrium points (a union of finitely many saddle connections).

We can also show that any cycle encloses an equilibrium point.

Finally, we can also extend Theorem 6.3 by inverting time, which amounts to replacing the ω-limit set $\mathcal{S}_\omega$ of the solution $x(t)$ by its α-limit set $\mathcal{S}_\alpha$. The region $\mathcal{M}$ is then backward invariant.

Example 1

Consider again the system

$$\begin{aligned}\dot{x}_1 &= -x_2 + \alpha x_1(x_1^2 + x_2^2 - 1)\\ \dot{x}_2 &= x_1 + \alpha x_2(x_1^2 + x_2^2 - 1)\end{aligned}$$

with $\alpha < 0$. Let $0 < \varepsilon \ll 1$ and let $\mathcal{M}$ be the annular region

$$\mathcal{M} = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid 1 - \varepsilon < x_1^2 + x_2^2 < 1 + \varepsilon\right\}.$$

Let

$$\begin{aligned}r &= \left(x_1^2 + x_2^2\right)^{1/2}\\ \varphi &= \arctan\left(\frac{x_2}{x_1}\right).\end{aligned}$$

Then

$$\mathcal{M} = \left\{(r, \varphi) \in \mathbb{R}_+ \times [0, 2\pi] \mid 1 - \varepsilon < r^2 < 1 + \varepsilon\right\}$$

and

$$\dot{r} = \alpha r(r^2 - 1).$$

We see that if $r < 1$, then $\dot{r} > 0$ and therefore all solutions that start from $0 < r(0) < 1 - \varepsilon$ must enter $\mathcal{M}$ at some time $t > 0$. Likewise, if $r > 1$, then $\dot{r} < 0$ and therefore all solutions that start from $r(0) > 1 + \varepsilon$ must enter $\mathcal{M}$ at some time $t' > 0$. Therefore $\mathcal{M}$ contains the ω-limit set of all solutions of the system (but the origin), it is thus forward invariant. Since it does not contain any equilibrium point, it must contain a periodic solution.

Example 2

Consider the system (from T. Kapitula, p. 54):

$$\begin{aligned}\dot{x}_1 &= \beta x_1 - x_2 + x_1(3x_1^2 + 2x_2^2)\\ \dot{x}_2 &= x_1 + \beta x_2 + x_2(3x_1^2 + 2x_2^2)\end{aligned}$$

with $\beta < 0$, and let us make a change of polar coordinates

$$\begin{aligned}r &= \left(x_1^2 + x_2^2\right)^{1/2}\\ \varphi &= \arctan\left(\frac{x_2}{x_1}\right).\end{aligned}$$

Then the system becomes

$$\begin{aligned}\dot{r} &= r\left(\beta + \left(2 + \cos^2(\varphi)\right) r^2\right)\\ \dot{\varphi} &= 1.\end{aligned}$$

Let $0 < \varepsilon \ll 1$ and let $\mathcal{M}$ be the annular region

$$\begin{aligned}\mathcal{M} &= \left\{(x_1, x_2) \in \mathbb{R}^2 \mid -\frac{\beta}{3-\varepsilon} < x_1^2 + x_2^2 < -\frac{\beta}{2+\varepsilon}\right\} \\ &= \left\{(r, \varphi) \in \mathbb{R}_+ \times [0, 2\pi] \mid -\frac{\beta}{3-\varepsilon} < r^2 < -\frac{\beta}{2+\varepsilon}\right\}.\end{aligned}$$

Since $r(\beta + 2r^2) < \dot{r} < r(\beta + 3r^2)$, $\mathcal{M}$ is backward invariant (the vector field points outwards on the boundary of $\mathcal{M}$). Since it does not contain any equilibrium point, it must contain a periodic solution.