# 7

# Bifurcations

## 7.1 Introduction

### 7.1.1 Definition

When analyzing the stability of nonlinear systems, we see that the same dynamical system can have very different asymptotic behaviors depending only on one parameter of the system, which we denote in this chapter by $\mu$. Moreover, the dynamical system has the same qualitative behavior for a range of values of this parameter, and switches suddenly between different qualitative behaviors at some particular values of this parameter. These values $\mu_0$ are called bifurcation points, and are defined below.

To make these parameters explicitly appear in the state equations, we recast equations (4.1) and (4.2) respectively as

$$\dot{x}(t) = F(x(t), \mu) \tag{7.1}$$

in continuous-time, and by

$$x(t+1) = F(x(t), \mu) \tag{7.2}$$

in discrete time. Function $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is at least a $C^1$-function, i.e. a continuously differentiable function.

**Definition 7.1.** *The system (7.1) or (7.2) undergoes a bifurcation at $\mu_0$, if there is no neighborhood $\mathcal{V}$ of $\mu_0$ on the real line $\mathbb{R}$ such that all systems with $\mu \in \mathcal{V}$ have the same qualitative behavior.*

The formulation of this definition is somewhat technical. The usual situation is that the system has a different qualitative behavior for $\mu < \mu_0$ and for $\mu > \mu_0$. The qualitative behavior for exactly $\mu = \mu_0$ may be one or the other, or something in between. For practical purposes the behavior at $\mu = \mu_0$ is not important.

A precise definition of what is meant by "two systems have the same qualitative behavior" can be given, but this goes beyond the scope of this introductory course. Basically, it means that there is a continuous coordinate and time transformation that transforms the solutions of one system to solutions of the other and vice versa. The notion is somewhat counterintuitive, because the qualitative behavior in a neighborhood of a stable node and a stable focus turn out to be the same. For more details, we refer to the literature on bifurcations, especially to the book by Y.Kuznetsov.

The definition is also applicable to systems that depend on more than one parameter, even though this case will not be discussed in detail here.

### 7.1.2 Example: Logistic Map

Let us consider the simple 1-dim discrete-time dynamical system is given by the iterations of an interval on the real line (Example 4.2.1), where the parameter $\mu$ is equal to $\lambda$ in (4.3), with $0 < \mu \leq 2$ so that the interval $[-1, 1]$ is invariant:

$$x(t+1) = 1 - \mu x^2(t). \tag{7.3}$$

We know the qualitative behavior of the system for parameter values $\mu$ close to $\mu_0$ from the analysis in Chapter 5, that (i) for $\mu < \mu_0 = 3/4$ there is one globally asymptotically stable fixed point. Its position changes continuously with $\mu$, whereas (ii) for $\mu > \mu_0 = 3/4$ the fixed point continues to exist and it still is a continuous function of $\mu$, even at $\mu_0$, but it is unstable. In addition, there is an asymptotically stable period 2-cycle. There is therefore a bifurcation at $\mu_0 = 3/4$, as shown in Figure 7.1.

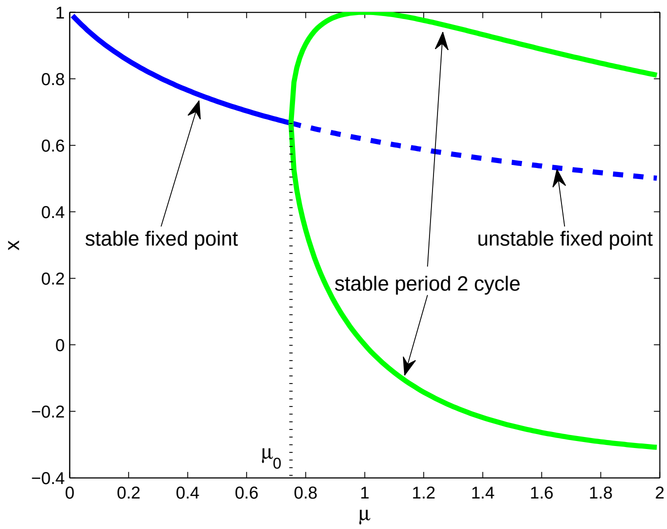


Figure 7.1: Bifurcation of a stable fixed point into an unstable fixed point and a stable 2-cycle for the logistic map, at $\mu_0 = 3/4$.

### 7.1.3 Local and global bifurcations

In the example of Section 7.1.2, the bifurcation at $\mu_0 = 3/4$ is actually a "gentle" transition. The period 2-cycle is born out of the fixed point at $\mu_0$ and continuously moves away from the fixed point as $\mu$ increases. Other bifurcations are much more abrupt changes of the asymptotic behavior. They are sometimes called *catastrophic*.

A different distinction of bifurcations is between *local* and *global* bifurcations. The bifurcation in Section 7.1.2 is local, because the qualitative change of asymptotic behavior takes place in a neighborhood of the fixed point. A global bifurcation of a 2-dimensional continuous time system is represented in Figure 7.2. The system has 2 equilibrium points that are saddle points. Locally around the equilibrium points, the asymptotic behavior does not change qualitatively. However, before the bifurcation (left) there are solutions that move in the horizontal direction from $-\infty$ to $+\infty$, but none that move from $+\infty$ to $-\infty$, whereas after the bifurcation (right), the situation is exactly the opposite. At the bifurcation (center), there is a solution that moves from the upper saddle point (at $t \to +\infty$) to the lower saddle point (at $t \to -\infty$). Such a solution that links two different equilibrium points is very special, and is called a *heteroclinic* solution (for an example of heteroclinic solution, consider the frictionless pendulum (see Exercise Sets), where the orbit from $(-\pi, 0)$ to $(\pi, 0)$ is a heteroclinic orbit). Correspondingly, the bifurcation is called a *heteroclinic bifurcation*.

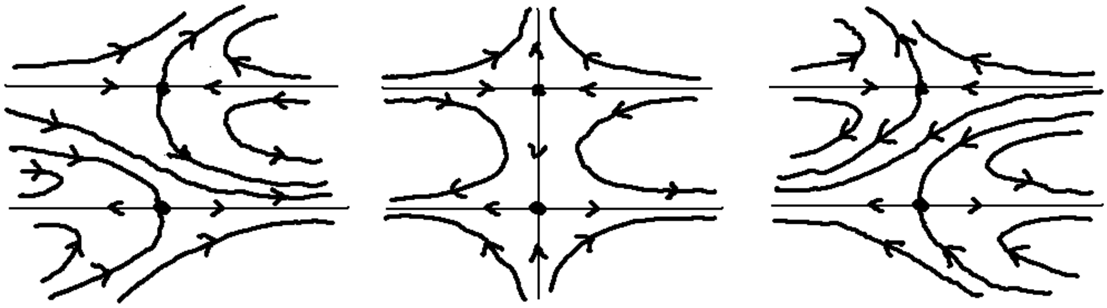

Figure 7.2: Flows of a 2-dimensional continuous-time system before (left), at (center) and after (right) a global bifurcation.

We shall not pursue global bifurcations any further and concentrate on local bifurcations of equilibrium and fixed points.

## 7.2 Bifurcations of equilibrium/fixed points

We consider continuous time systems of the form (7.1) with an equilibrium point $\overline{x}_0$ when the parameter has the value $\mu_0$, i.e. such that

$$F(\overline{x}_0, \mu_0) = 0 \tag{7.4}$$

and discrete-time systems of the form (7.2) with a fixed point $\overline{x}_0$ when the parameter has the value $\mu_0$, i.e. such that

$$F(\overline{x}_0, \mu_0) = \overline{x}_0. \tag{7.5}$$

We expect that for parameters in a neighborhood of $\mu_0$ there is exactly one equilibrium/fixed point in a neighborhood of $\overline{x}_0$ and that this equilibrium/fixed point is a continuously differentiable function of the parameter. This is usually the case and can be proved by the implicit function theorem, which a classic theorem that can be found in any book on multivariate analysis (multivariable calculus), and which we recall here.

**Theorem 7.1** (Implicit Function Theorem)**.** *Let* $F : \mathbb{R}^{n+m} \to \mathbb{R}^n$ *be a* $C^1$*-function and suppose that*

$$F(x_0, y_0) = 0 \tag{7.6}$$

*with* $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$*. Suppose that the* $n \times n$ *Jacobian matrix of* $F$ *with respect to* $x$ *is*

$$J_x(x_0, y_0) = \frac{\partial F}{\partial x}(x_0, y_0) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(x_0, y_0) & \frac{\partial F_1}{\partial x_2}(x_0, y_0) & \dots & \frac{\partial F_1}{\partial x_n}(x_0, y_0) \\ \frac{\partial F_2}{\partial x_1}(x_0, y_0) & \frac{\partial F_2}{\partial x_2}(x_0, y_0) & \dots & \frac{\partial F_2}{\partial x_n}(x_0, y_0)) \\ \vdots & & \ddots & \\ \frac{\partial F_n}{\partial x_1}(x_0, y_0) & \frac{\partial F_n}{\partial x_2}(x_0, y_0) & \dots & \frac{\partial F_n}{\partial x_n}(x_0, y_0) \end{bmatrix} \tag{7.7}$$

*is non-singular (i.e is invertible). Then there is a neighborhood* $\mathcal{U}$ *of* $(x_0, y_0)$ *in* $\mathbb{R}^{n+m}$*, a neighborhood* $\mathcal{V}$ *of* $y_0$ *in* $\mathbb{R}^m$ *and a* $C^1$*- function* $g : \mathcal{V} \to \mathbb{R}^n$ *such that all solutions of* $F(x, y) = 0$ *in* $\mathcal{U}$ *are given by* $x = g(y)$*. Moreover,*

$$\begin{aligned} \frac{\partial g}{\partial y}(y_0) &= -\left(\frac{\partial F}{\partial x}\right)^{-1}(x_0, y_0) \cdot \frac{\partial F}{\partial y}(x_0, y_0) \\ &= -J_x^{-1}(x_0, y_0) J_y(x_0, y_0). \end{aligned} \tag{7.8}$$

We apply this theorem to the *equilibrium point equation* for continuous-time systems

$$F(\overline{x}, \mu) = 0 \tag{7.9}$$

given (7.4), and to the *fixed point equation* for discrete-time systems

$$F(\overline{x}, \mu) - \overline{x} = 0 \tag{7.10}$$

given (7.5). The conclusion is that if the Jacobian matrix (7.7) does not have the eigenvalue 0 in the case of a continuous time system (respectively, the eigenvalue 1 in the case of a discrete time system) in a neighborhood of $(\overline{x}_0, \mu_0)$, the equilibrium/fixed points are given by a continuously differentiable 1-parameter family $\overline{x}(\mu)$ with

$$\overline{x}(\mu_0) = \overline{x}_0 \tag{7.11}$$

and

$$\frac{\partial \overline{x}}{\partial \mu}(\mu_0) = -\left(\frac{\partial F}{\partial x}\right)^{-1}(\overline{x}_0, \mu_0) \cdot \frac{\partial F}{\partial \mu}(\overline{x}_0, \mu_0). \tag{7.12}$$

The fact that the equilibrium/fixed point not only exists at $\mu = \mu_0$ but also in a neighborhood of this value does not yet mean that there is no bifurcation at $\mu_0$. However, it can be shown that if the Jacobian matrix is hyperbolic, i.e. if it has no eigenvalue on the imaginary axis for continuous-time systems (resp., no eigenvalue on the unit circle for discrete-time systems), then there is no bifurcation at $\mu_0$.

Hence, in the case of continuous-time systems, equilibrium points can only undergo local bifurcations if the Jacobian matrix at the bifurcation point has an eigenvalue on the imaginary axis. Two cases have to be distinguished, as shown in Figure 7.3:

- a zero eigenvalue: the corresponding generic bifurcation is the *fold bifurcation* (sometimes also called saddle-node or tangent bifurcation); if more degeneracy or symmetry conditions apply, the bifurcation can be e.g., a transcritical or pitchfork bifurcation (see later),
- two complex conjugate eigenvalues: the corresponding generic bifurcations is the *Andronov-Hopf bifurcation.*

"Generic" means that if system parameters were chosen randomly (but observing the bifurcation constraints) then this type of bifurcation would be obtained. Other types are possible, but they require additional constraints or would have a zero-probability to be obtained with randomly chosen parameters. We will see for example the *pitchfork bifurcation*, which corresponds also to a zero eigenvalue of the Jacobian, but for an odd function $F(x, \mu)$ of $x$. Another example is the *transcritical bifurcation*, valid for instance for systems where $F(x, \mu)$ is an even function of $\mu$.

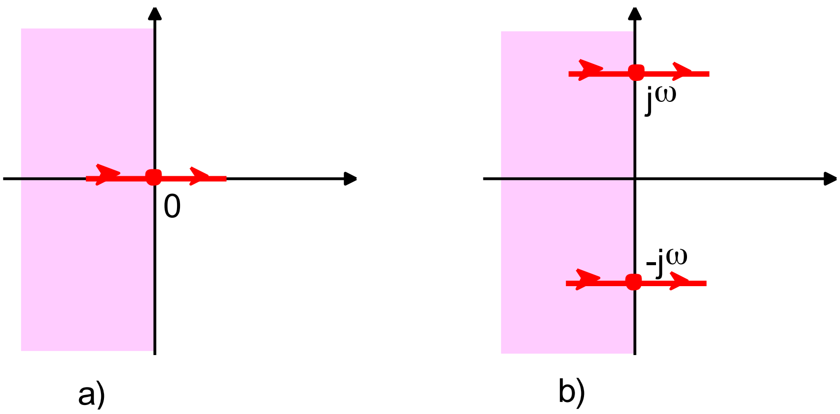


Figure 7.3: Simplest types of bifurcations of equilibrium points of continuous time systems, when a) a single eigenvalue, or b) a pair of complex conjugate eigenvalues cross the imaginary axis. The corresponding generic bifurcations are a) the Fold and b) the Andronov-Hopf bifurcations.

In the case of discrete-time systems, fixed points can only undergo local bifurcations if the Jacobian matrix at the bifurcation point has an eigenvalue on the unit circle. Three cases have to be distinguished, as shown in Figure 7.4:

- the eigenvalue $+1$: the corresponding generic bifurcation is the *fold bifurcation*,
- the eigenvalue $-1$: the corresponding generic bifurcation is the *flip bifurcation* (sometimes also called period-doubling bifurcation),
- two complex conjugate eigenvalues: the corresponding generic bifurcations is the *Neimark-Sacker bifurcation.*

The exact conditions for the various bifurcations will be given in the next sections. We will consider them in the lowest possible dimension of state space, i.e. in dimension 1 for the fold, transcritical and flip bifurcations and in dimension 2 for the Andronov-Hopf bifurcation.

## 7.3 Fold (or Saddle-Node) bifurcation of equilibrium/fixed points in 1-dim. state-space

Consider (7.1) with $n = 1$, i.e.,

$$\dot{x} = F(x, \mu), \tag{7.13}$$

with $F : \mathbb{R}^2 \to \mathbb{R}$ with

$$F(\overline{x}_0, \mu_0) = 0 \tag{7.14}$$

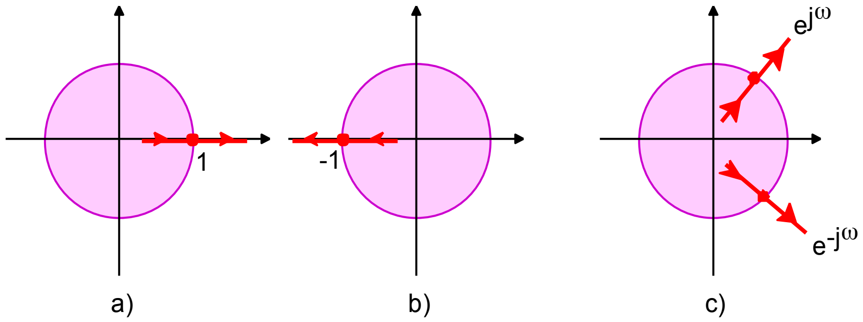


Figure 7.4: Simplest types of bifurcations of fixed points of continuous time systems, when a) a single eigenvalue crosses the unit circle at 1 or b) at -1 or c) when a pair of complex conjugate eigenvalues cross the unit circle. The corresponding generic bifurcations are a) the Fold, b) the Flip and c) the Neimark-Sacker bifurcations.

because of (7.4) and

$$\frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) = 0 \tag{7.15}$$

because of the zero eigenvalue of the Jacobian matrix at $(\overline{x}_0, \mu_0)$.

Therefore the Taylor expansion of $F$ around $(\overline{x}_0, \mu_0)$ up to the lowest order non-vanishing terms is

$$F(x, \mu) \approx a(\mu - \mu_0) + b(x - \overline{x}_0)^2, \tag{7.16}$$

where the higher order terms are neglected, and where

$$\begin{aligned} a &= \frac{\partial F}{\partial \mu}(\overline{x}_0, \mu_0) \\ b &= \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0). \end{aligned}$$

In the generic case where $a, b \neq 0$, indeed these are the lowest order non-vanishing terms. Note that we do not consider the terms proportional to $(x - \overline{x}_0)(\mu - \mu_0)$, nor to $(\mu - \mu_0)^2$, because they are dominated by the term $a(\mu - \mu_0)$. By neglecting the higher order terms of the Taylor expansion, the system (7.13) becomes therefore

$$\dot{x} = a(\mu - \mu_0) + b(x - \overline{x}_0)^2. \tag{7.17}$$

Setting $\dot{x} = 0$ in the previous equation, we obtain the equilibrium point equation (7.9), which has the solution

$$\mu = \mu_0 - \frac{b}{a}\left(\overline{x} - \overline{x}_0\right)^2, \tag{7.18}$$

or equivalently,

$$\overline{x} = \overline{x}_0 \pm \sqrt{\frac{a}{b}\left(\mu_0 - \mu\right)}. \tag{7.19}$$

We see that the existence of the equilibrium points in the vicinity of $(\overline{x}_0, \mu_0)$ depends on the sign of $a/b$.

These equilibrium points are hyperbolic for $\mu \neq \mu_0$ and their stability can be deduced from the sign of

$$J_x(\overline{x}, \mu(\overline{x})) = \frac{\partial F}{\partial x}(\overline{x}, \mu(\overline{x})) = 2b\left(\overline{x} - \overline{x}_0\right). \tag{7.20}$$

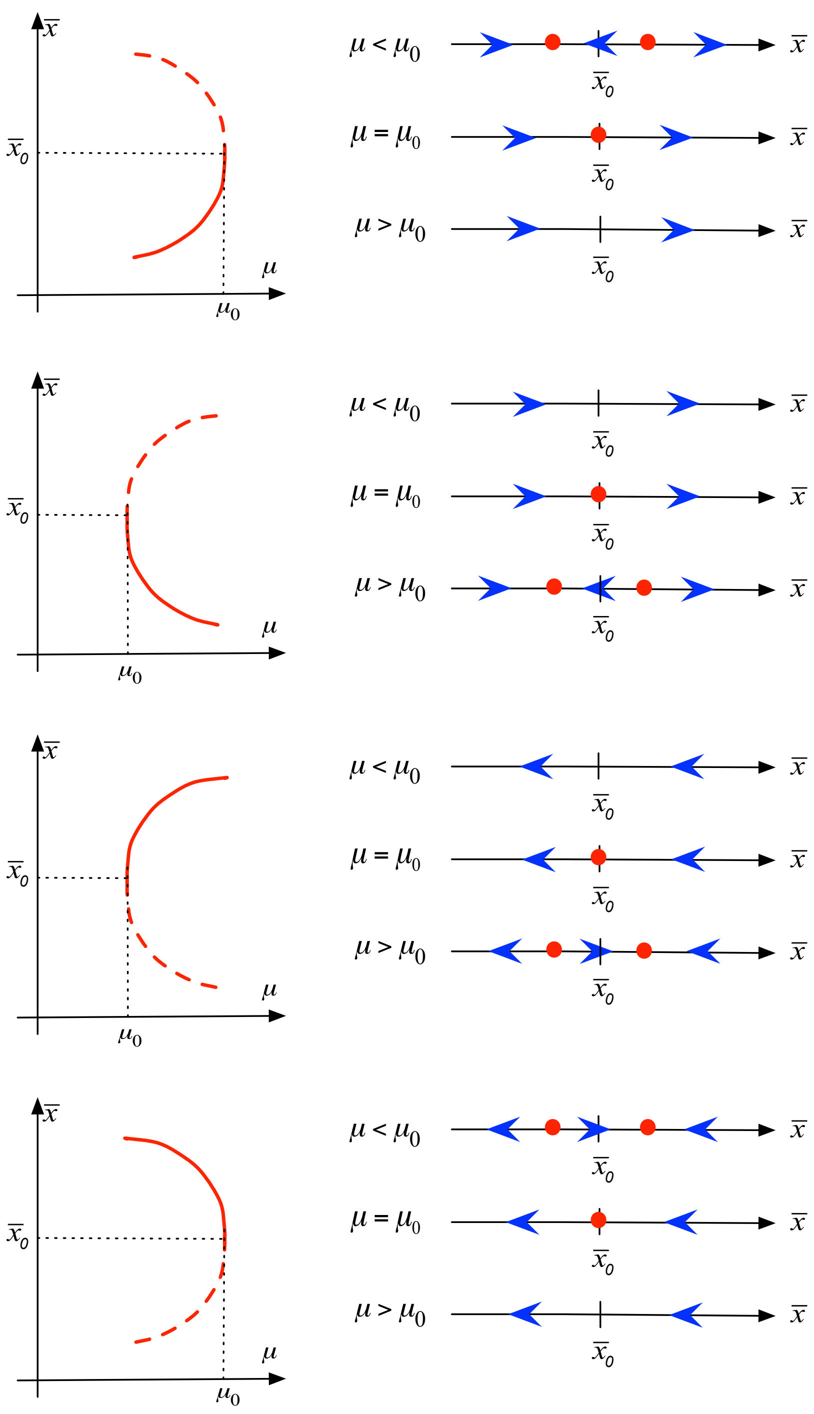

Figure 7.5: Fold bifurcation of a continuous-time system (7.13), with, from top to bottom: $a > 0, b > 0$ (top), $a < 0, b > 0$ (second from top), $a > 0, b < 0$ (third from top) and $a < 0, b < 0$ (bottom). On the left, bifurcation diagram representing the family of equilibrium points in a neighborhood of $(\overline{x}_0, \mu_0)$. The solid line is asymptotically stable equilibrium, the dotted line is the unstable equilibrium point. On the right, dynamics in the 1-dimensional state space.

Depending on the signs of $a$ and $b$, we can consider 4 cases, which are represented in Figure 7.5.

The analysis of the fold bifurcation in the case of discrete time systems is similar, and left to the reader. The following theorem states that the higher order terms in (7.16) do not change the qualitative nature of the asymptotic behavior of the solutions in a neighborhood of $(\overline{x}_0, \mu_0)$. In fact, a parameter-dependent coordinate transformation is able to eliminate them.

**Theorem 7.2** (Fold Bifurcation in 1-dim systems). *Let the continuous-time (respectively, discrete-time) system given by*

$$\dot{x}(t) = F(x(t), \mu),$$

*respectively by*

$$x(t+1) = F(x(t), \mu),$$

*and* $F : \mathbb{R}^2 \to \mathbb{R}$ *be* $C^2$*-function (twice continuously differentiable). Let* $\overline{x}_0 \in \mathbb{R}$ *and* $\mu_0 \in \mathbb{R}$ *be such that*

$$\begin{aligned} F(\overline{x}_0, \mu_0) &= 0 \qquad (\textit{ resp.}, \ = \overline{x}_0) \\ \frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) &= 0 \qquad (\textit{ resp.}, \ = 1) \\ \frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0) &\neq 0 \\ \frac{\partial F}{\partial \mu}(\overline{x}_0, \mu_0) &\neq 0. \end{aligned}$$

*Then the system undergoes a fold (also called saddle-node) bifurcation at* $(\overline{x}_0, \mu_0)$. *That is, in a neighborhood of* $(\overline{x}_0, \mu_0)$:

*(i) for* $\mu < \mu_0$, *there are two equilibrium/fixed points, one asymptotically stable, the other unstable, and for* $\mu > \mu_0$ *there is none, or vice-versa;*

*(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is*

$$\dot{x}(t) = \mu \pm x^2(t) \tag{7.21}$$

*for a continuous-time system, or*

$$x(t+1) = \mu + x(t) \pm x^2(t). \tag{7.22}$$

*for a discrete-time system.*

## 7.4 Transcritical bifurcation of equilibrium/fixed points in 1-dim. state-space

Suppose that at an equilibrium (respectively, fixed) point $\overline{x}_0$ with a Jacobian matrix with eigenvalue 0 (respectively, 1), i.e.

$$\frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) = 0 \qquad (\text{ resp.}, \ = 1) \tag{7.23}$$

we have also that

$$\frac{\partial F}{\partial \mu}(\overline{x}_0, \mu_0) = 0. \tag{7.24}$$

Such is the case for example if $\mu_0 = 0$ and if $F$ is an even function in $\mu$. Then, the Taylor expansion of $F$ around $(\overline{x}_0, \mu_0)$ up to the lowest order non-vanishing terms has the form

$$F(x, \mu) \approx a(x - \overline{x}_0)^2 + b(\mu - \mu_0)(x - \overline{x}_0) + c(\mu - \mu_0)^2, \tag{7.25}$$

for a continuous-time system, where

$$\begin{aligned} a &= \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0) \\ b &= \frac{\partial^2 F}{\partial \mu \partial x}(\overline{x}_0, \mu_0) \\ c &= \frac{1}{2}\frac{\partial^2 F}{\partial \mu^2}(\overline{x}_0, \mu_0), \end{aligned}$$

with a similar expression for discrete-time systems. Apart from the higher order terms, (7.25) is a pure quadratic form. If it is positive or negative definite, the equilibrium point is isolated and the behavior in a neighborhood of $(\overline{x}_0, \mu_0)$ is trivial: $\dot{x} > 0$ for all $\mu \neq \mu_0$ or $\dot{x} < 0$ for all $\mu \neq \mu_0$. If the quadratic form is not positive or negative definite, there are two lines that cross at $(\overline{x}_0, \mu_0)$ on which $F$ vanishes. Up to second order approximation, these are straight lines. By a linear coordinate transformation, we can make one of them parallel to the $x$-axis. In the new coordinates, the Taylor expansion of $F$ around $(\overline{x}_0, \mu_0)$ up to the lowest order non-vanishing terms has the form

$$F(x, \mu) \approx a(\mu - \mu_0)(x - \overline{x}_0) + b(x - \overline{x}_0)^2$$

and we examine the system in the new coordinates as

$$\dot{x} = a(x - \overline{x}_0)^2 + b(\mu - \mu_0)(x - \overline{x}_0). \tag{7.26}$$

with a similar reasoning for discrete-time systems.

Setting $\dot{x} = 0$ in (7.26), we find that the the equilibrium points are

$$\overline{x}(\mu) = \overline{x}_0 \tag{7.27}$$

$$\overline{x}(\mu) = \overline{x}_0 - \frac{b}{a}(\mu - \mu_0). \tag{7.28}$$

They are hyperbolic for $\mu \neq \mu_0$. For continuous time systems, the equilibrium point is asymptotically stable if and only if

$$J_x(\overline{x}, \mu) = \frac{\partial F}{\partial x}(\overline{x}, \mu) = 2a(\overline{x} - \overline{x}_0) + b(\mu - \mu_0) < 0,$$

which becomes $b(\mu - \mu_0) < 0$ for $\overline{x}(\mu)$ given by (7.27) and $-b(\mu - \mu_0) < 0$ for $\overline{x}(\mu)$ given by (7.28). Therefore, for all choices of signs for $a$ and $b$ the qualitative behavior is the same, namely one equilibrium point is asymptotically stable and one is unstable, but at the bifurcation point they exchange their stability. The sign of $b$ only decides which is stable at which side of the bifurcation point, and the sign of $a$ decides whether the slope of the second equilibrium point is positive or negative. Figure 7.6 shows the bifurcation diagram for the case $a > 0$, $b > 0$.

The analysis of the transcritical bifurcation in the case of discrete time systems is similar, and left to the reader. The following theorem states that the higher order terms in (7.35) do not change the qualitative nature of the asymptotic behavior of the solutions in a neighborhood of $(\overline{x}_0, \mu_0)$. In fact, a parameter-dependent coordinate transformation is able to eliminate them.

**Theorem 7.3** (Transcritical Bifurcation in 1-dim systems)**.** *Let the continuous-time (respectively, discrete-time) system given by*

$$\dot{x}(t) = F(x(t), \mu),$$

*respectively by*

$$x(t+1) = F(x(t), \mu),$$

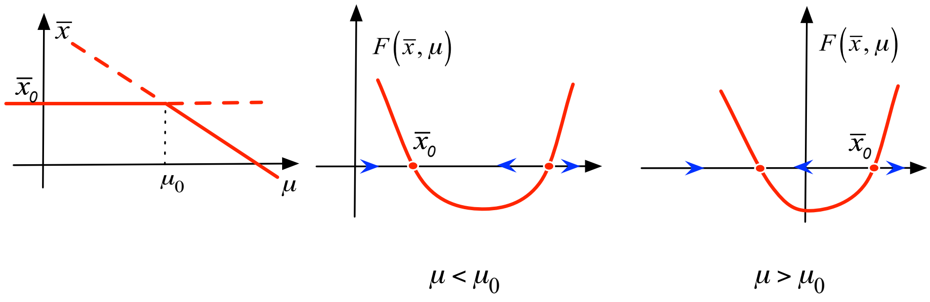


Figure 7.6: Transcritical bifurcation of a continuous-time system with $a > 0, b > 0$. On the left, bifurcation diagram representing the family of equilibrium points in a neighborhood of $(\overline{x}_0, \mu_0)$. The solid line is the asymptotically stable equilibrium, the dotted line is the unstable equilibrium point(s). On the right, dynamics in the 1-dimensional state space for $\mu < \mu_0$ and $\mu > \mu_0$.

*and let $F : \mathbb{R}^2 \to \mathbb{R}$ be a $C^2$-function (two times continuously differentiable). $F : \mathbb{R}^2 \to \mathbb{R}$ be $C^2$-function (twice continuously differentiable). Let $\overline{x}_0 \in \mathbb{R}$ and $\mu_0 \in \mathbb{R}$ be such that*

$$\begin{aligned}
F(\overline{x}_0, \mu_0) &= 0 \qquad (\ \textit{resp.}, \ = \overline{x}_0) \\
\frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) &= 0 \qquad (\ \textit{resp.}, \ = 1) \\
\frac{\partial F}{\partial \mu}(\overline{x}_0, \mu_0) &= 0 \\
\frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0) &\neq 0 \\
\left[\frac{\partial^2 F}{\partial \mu \partial x}(\overline{x}_0, \mu_0)\right]^2 - \frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0)\frac{\partial^2 F}{\partial \mu^2}(\overline{x}_0, \mu_0) &> 0.
\end{aligned}$$

*Then the system undergoes a transcritical (also called saddle-node) bifurcation at $(\overline{x}_0, \mu_0)$. That is, in a neighborhood of $(\overline{x}_0, \mu_0)$:*

*(i) for $\mu \neq \mu_0$, there are two equilibrium/fixed points, one asymptotically stable, the other unstable. They switch stability at $\mu = \mu_0$;*

*(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is*

$$\dot{x}(t) = \mu x(t) \pm x^2(t) \tag{7.29}$$

*for a continuous-time system, or*

$$x(t+1) = (1+\mu)x(t) \pm x^2(t). \tag{7.30}$$

*for a discrete-time system.*

## Example: SIS Epidemic Model

Suppose a virus propagates in a population whose members are divided in two classes: the individuals who are *susceptible* (state S) of being contaminated by the virus but are still healthy, or the individuals who are *infected* (state I) with the virus. An individual who is infected recover after some amount of time and switch back to the S state, until it gets contaminated again and becomes I again: this model is known as the SIS epidemic model.

In general, epidemics propagate on a network, whose nodes are the individuals and whose edges represent the possible direct contacts between individuals. The study of epidemic propagation on arbitrary contact networks is a very challenging problem, with many open challenges to solve. One common but strong approximation is to assume a complete contact network (with an edge between every pair of nodes), where every individual can be in contact any other individual. In this case, the model can be well approximated by a set of coupled nonlinear ordinary differential equations using mean field techniques, these are the so-called population models. We consider closed models, where the population size remains constant at all times.

Let $\beta > 0$ be the infection rate, which is the product of the average number of contacts that each individual has per time unit by the probability of infection per contact. Each infected individual contaminates therefore a proportion $\beta S(t)$ of the total population per time unit. , where $S(t) = 1$ is the fraction of susceptible individuals. Let $I(t) = 1 - S(t)$ be the fraction of the population that is infected at time $t$. The rate at which $S(t)$ decreases (or equivalently, $I(t)$ grows) is then equal to $\beta S(t)I(t)$, since there is a proportion of $I(t)$ individuals infected at time $t$.

Infected individuals recover however at a constant rate per individual, which we denote by $\gamma > 0$ (hence $1/\gamma$ would be the average duration of the infectious period), and as result the recovering process contributes to a growth of $S(t)$ grows (or equivalently, a decrease of $I(t)$) equal to $\gamma I(t)$ per time unit. The resulting continuous-time model is therefore

$$\frac{dS}{dt}(t) = -\beta S(t)I(t) + \gamma I(t) \tag{7.31}$$

$$\frac{dI}{dt}(t) = \beta S(t)I(t) - \gamma I(t) \tag{7.32}$$

Clearly these two equations can be reduced to a single o.d.e. by eliminating $S(t) = 1 - I(t)$, to become

$$\frac{dI}{dt}(t) = \beta(1 - I(t))I(t) - \gamma I(t) \tag{7.33}$$

This dynamical system has two equilibrium points, $I_h^\star = 0$, where all the population is healthy and the virus dies out, and $I_e^\star = 1 - \gamma/\beta$, which is the endemic equilibrium point and which is positive only if $\beta > \gamma$. Let us denote $R = \beta/\gamma$. This ratio $R$ between the infection rate and healing rate is known in epidemiology as the *basic reproductive number*. One easily checks that $I_h^\star = 0$ is asymptotically stable if $R < 1$ but becomes unstable if $R > 1$ , while $I_e^\star$ is then asymptotically stable. At $R = 1$, the two equilibria collide and switch stability, and the SIS epidemic system undergoes a transcritical bifurcation.

## 7.5 Pitchfork bifurcation of equilibrium/fixed points in 1-dim. state-space

When the Jacobian matrix at the equilibrium (respectively, fixed) point has the eigenvalue 0 (resp., 1) at the bifurcation parameter value $\mu_0$, the fold bifurcation ($a \neq 0$ and $b \neq 0$) is the generic case. If there are symmetries present, or other special constraints, we may have $a = 0$ and/or $b = 0$. A common symmetry constraint is that $F$ is an odd function in $x$, i.e. that for all $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$,

$$F(-x, \mu) = -F(x, \mu). \tag{7.34}$$

This implies in particular that $F(0, \mu) = 0$ and thus that $\overline{x} = 0$ is an equilibrium (respectively, fixed) point for all $\mu \in \mathbb{R}$. Furthermore, the Taylor expansion of $F(x, \mu)$ around $(0, \mu_0)$ has no even terms in $x$. The first terms are

$$F(x, \mu) \approx a(\mu - \mu_0)x + bx^3, \tag{7.35}$$

for a continuous-time system, where

$$\begin{aligned} a &= \frac{\partial F^2}{\partial \mu \partial x}(0, \mu_0) \\ b &= \frac{1}{6}\frac{\partial^3 F}{\partial x^3}(0, \mu_0). \end{aligned}$$

Similarly, for a discrete-time system, the first terms of the Taylor expansion are

$$F(x, \mu) \approx x + a(\mu - \mu_0)x + bx^3, \tag{7.36}$$

Note that the absence of the term $x$ in (7.35) and its presence in (7.36) is a consequence of the eigenvalue 0, resp. 1, of the Jacobian matrix at $(0, \mu_0)$.

Let us neglect the higher order terms and consider the continuous-time system given by

$$\dot{x} = a(\mu - \mu_0)x + bx^3. \tag{7.37}$$

Setting $\dot{x} = 0$ in the previous equation, we obtain the equilibrium point equation (7.9), which reads

$$0 = \overline{x}\left(a(\mu - \mu_0) + b\overline{x}^2\right). \tag{7.38}$$

As already noted, there is the equilibrium point 0 for all $\mu \in \mathbb{R}$. For $\mu \neq \mu_0$, it is hyperbolic and it is asymptotically stable if and only if

$$\frac{\partial F}{\partial x}(0, \mu) = a(\mu - \mu_0) < 0.$$

In addition, there is a second family of equilibrium points for $a(\mu - \mu_0) + b\overline{x}^2 = 0$, which are

$$\mu = \mu_0 - \frac{b}{a}\overline{x}^2. \tag{7.39}$$

Again, for $\mu \neq \mu_0$, it is hyperbolic and it is asymptotically stable if and only if

$$\frac{\partial F}{\partial x}(\overline{x}, \mu) = \frac{\partial F}{\partial x}\left(\overline{x}, \mu_0 - \frac{b}{a}\overline{x}^2\right) = 2b\overline{x}^2 < 0.$$

Depending on the signs of $a$ and $b$, we can consider again 4 cases. The first two are represented in Figure 7.7, the two others are similar. Observe that the first pitchfork bifurcation (with $a > 0, b > 0$) is subcritical, whereas the second one (with $a > 0, b < 0$) is supercritical.

The analysis of the pitchfork bifurcation in the case of discrete time systems is similar, and left to the reader. The following theorem states that the higher order terms in (7.35) do not change the qualitative nature of the asymptotic behavior of the solutions in a neighborhood of $(\overline{x}_0, \mu_0)$. In fact, a parameter-dependent coordinate transformation is able to eliminate them.

**Theorem 7.4** (Pitchfork Bifurcation in 1-dim systems)**.** *Let the continuous-time (respectively, discrete-time) system given by*

$$\dot{x}(t) = F(x(t), \mu),$$

*respectively by*

$$x(t+1) = F(x(t), \mu),$$

*and let $F : \mathbb{R}^2 \to \mathbb{R}$ be a $C^3$-function (three times continuously differentiable), which is odd (i.e. verifies (7.34)). Let $\mu_0 \in \mathbb{R}$ be such that*

$$\begin{aligned} \frac{\partial F}{\partial x}(0, \mu_0) &= 0 \qquad (\textit{ resp.}, \ = 1) \\ \frac{\partial^2 F}{\partial x \partial \mu}(0, \mu_0) &\neq 0 \\ \frac{\partial^3 F}{\partial x^3}(0, \mu_0) &\neq 0. \end{aligned}$$

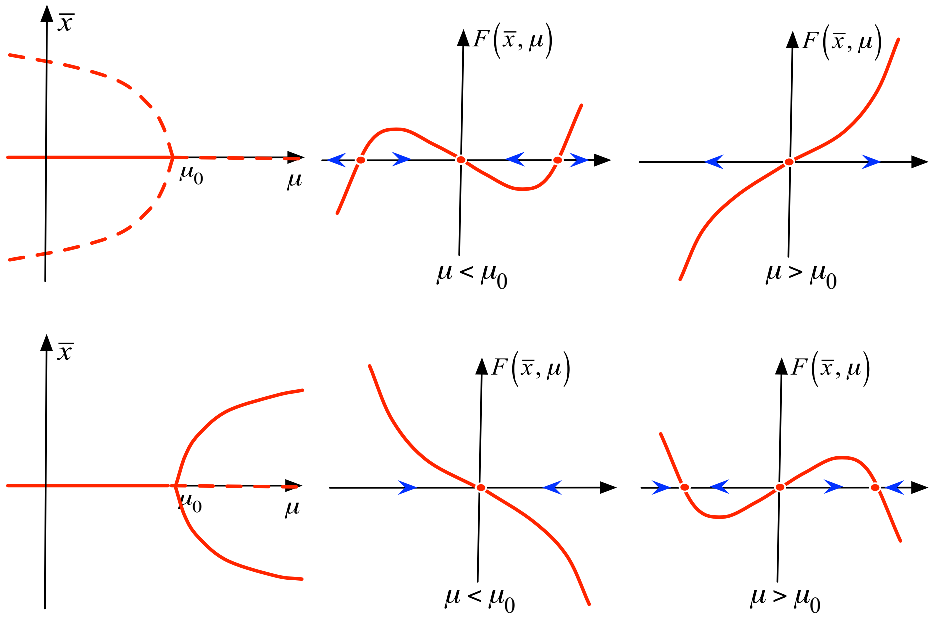


Figure 7.7: Pitchfork bifurcation of a continuous-time system with $a > 0, b > 0$ (top) and $a > 0, b < 0$ (bottom). On the left, bifurcation diagram representing the family of equilibrium points in a neighborhood of $(\overline{x}_0, \mu_0)$. The solid line is the asymptotically stable equilibrium, the dotted line is the unstable equilibrium point(s). On the right, dynamics in the 1-dimensional state space for $\mu < \mu_0$ and $\mu > \mu_0$.

*Then the system undergoes a pitchfork bifurcation at $(0, \mu_0)$, that is, in a neighborhood of $(\overline{x}_0, \mu_0)$,*

*(i) for $\mu < \mu_0$, the origin is the only equilibrium/fixed point and it is asymptotically stable, whereas for $\mu > \mu_0$ the origin is an unstable equilibrium/fixed point, and in addition, there are two asymptotically stable equilibrium/fixed points, or vice-versa (this is called a* supercritical pitchfork bifurcation*) or for $\mu < \mu_0$, the origin is an asymptotically stable equilibrium/fixed point and in addition there are two unstable equilibrium/fixed points, whereas for $\mu > \mu_0$ the origin is the only equilibrium/fixed point and it is unstable, or vice-versa (this is called a* subcritical pitchfork bifurcation*);*

*(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is*

$$\dot{x}(t) = \mu x(t) \pm x^3(t) \tag{7.40}$$

*for a continuous-time system, or*

$$x(t+1) = (1+\mu)x(t) \pm x^3(t) \tag{7.41}$$

*for a discrete-time system.*

## 7.6 Flip bifurcation of fixed points in 1-dim. state-space

The flip bifurcation in 1-dim state-space is proper to discrete-time systems. Suppose that at a fixed point $\overline{x}_0$ of a discrete time system the Jacobian matrix has the eigenvalue -1 for the parameter value $\mu = \mu_0$. Then, by the implicit function theorem, the solutions of the fixed point equation (7.10) in a neighborhood of $(\overline{x}_0, \mu_0)$ is a continuously differentiable 1-parameter family $\overline{x}(\mu)$ with $\overline{x}(\mu_0) = \overline{x}_0$. It can be shown that by a parameter-dependent continuous coordinate transformation, the system can be brought to its normal form $x(t+1) = F(x(t), \mu)$ with

$$F(x, \mu) = -(1+\mu)x \pm x^3 \tag{7.42}$$

Now, iterate $F$ once more, and call the resulting function $F^{(2)}(x, \mu)$:

$$\begin{aligned} F^{(2)}(x,\mu) &= F(F(x,\mu)) = -(1+\mu)\left(-(1+\mu)x \pm x^3\right) \pm \left(-(1+\mu)x \pm x^3\right)^3 \\ &= (1+\mu)^2 x \mp (1+\mu)\left(2+2\mu+\mu^2\right)x^3 + O\left(x^5\right). \end{aligned}$$

We see that $F^{(2)}(x, \mu)$ is an odd function of $x$ (and thus $F^{(2)}(0, 0) = 0$), with

$$\begin{aligned} \frac{\partial F^{(2)}}{\partial x}(0,0) &= 1 \\ \frac{\partial^2 F^{(2)}}{\partial x \partial \mu}(0,0) &= 2 \neq 0 \\ \frac{\partial^3 F^{(2)}}{\partial x^3}(0,0) &= \pm 12 \neq 0. \end{aligned}$$

Therefore, Theorem 7.4 indicates that the fixed point 0 of the system

$$x(t+1) = F^{(2)}(x(t), \mu) \tag{7.43}$$

undergoes a pitchfork bifurcation for $\mu = 0$, which is supercritical if the sign in (7.42) is + and subcritical if it is $-$.

Now, any fixed point of the system (7.43) is either a fixed point or a period 2 cycle for the original system (7.42). Since we know that for a given $\mu$ there is only one fixed point, namely $\overline{x} = 0$, we conclude

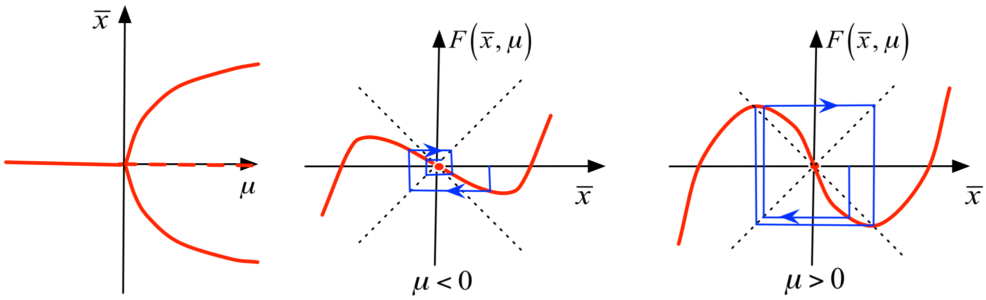


Figure 7.8: Supercritical flip bifurcation of a discrete-time system. On the left, bifurcation diagram representing the stable (solid line) or unstable (dashed line) equilibrium point the origin, and the stable 2-cycle (solid curves). On the right, dynamics in the 1-dimensional state space for $\mu < 0$ and $\mu > 0$.

that whenever $F^{(2)}$ has simultaneously other fixed points, they must constitute a period 2 cycle. Figure 7.8 shows a supercritical flip bifurcation, derived from a supercritical pitchfork bifurcation.

As in the other types of bifurcations, the following theorem states that the higher order terms in (7.42) do not change the qualitative nature of the asymptotic behavior of the solutions in a neighborhood of $(\overline{x}_0, \mu_0)$.

**Theorem 7.5** (Flip Bifurcation in 1-dim systems). *Let the discrete-time system*

$$x(t+1) = F(x(t), \mu),$$

*given by the $C^3$-function $F : \mathbb{R}^2 \to \mathbb{R}$. Let $\overline{x}_0 \in \mathbb{R}$ and $\mu_0 \in \mathbb{R}$ be such that*

$$\begin{aligned} F(\overline{x}_0, \mu_0) &= \overline{x}_0 \\ \frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) &= -1 \\ \left[\frac{\partial^2 F}{\partial \mu \partial x} + \frac{1}{2}\left(\frac{\partial F}{\partial \mu}\right)\left(\frac{\partial^2 F}{\partial x^2}\right)\right](\overline{x}_0, \mu_0) = \alpha &\neq 0 \\ \frac{1}{6}\frac{\partial^3 F}{\partial x^3}(\overline{x}_0, \mu_0) + \left(\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\overline{x}_0, \mu_0)\right)^2 = \beta &\neq 0. \end{aligned}$$

*Then the system undergoes a flip bifurcation at $(\overline{x}_0, \mu_0)$, that is, in a neighborhood of $(\overline{x}_0, \mu_0)$,*

*(i) for $\mu < \mu_0$, there is an asymptotically stable fixed point, whereas for $\mu > \mu_0$ the fixed point is unstable, and in addition, there is an asymptotically stable 2-cycle, or vice-versa (this is called a* supercritical flip bifurcation*) or for $\mu < \mu_0$, there is an asymptotically stable fixed point and an unstable 2-cycle, whereas for $\mu > \mu_0$ there is only the fixed point and it is unstable, or vice-versa (this is called a* subcritical flip bifurcation*);*

*(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is*

$$x(t+1) = -(1+\mu)x(t) \pm x^3(t). \tag{7.44}$$

Depending on the signs of $\alpha$ and $\beta$ in the theorem, we can consider 4 cases, similarly to previous bifurcation types. For instance, the case $\alpha > 0, \beta < 0$ occurs leads to the supercritical flip bifurcation represented in Figure 7.5.

### Example: Logistic Map

The fixed point of the logistic map (7.3) looses its stability at $\mu = \mu_0 = 3/4$ and at the same time a stable period 2 periodic solution appears. At $\mu_0 = 3/4$, one can compute that with $F(x, \mu) = 1 - \mu x^2$, and with $\overline{x}_0 = F(\overline{x}_0, 3/4) = 2/3$ denoting the fixed point,

$$\begin{aligned} \frac{\partial F}{\partial x}(\overline{x}_0, \mu_0) &= -2\mu_0 \overline{x}_0 = -1 \\ \alpha &= \left[-2 + \frac{1}{2}(-\overline{x}_0^2)(-2)\right] = -14/9 \neq 0 \\ \beta &= 0 + \left(\frac{1}{2}(-2)\right)^2 = 1 \neq 0. \end{aligned}$$

Therefore the system undergoes a supercritical flip bifurcation at $\mu = 3/4$.

## 7.7 Andronov-Hopf bifurcation of equilibrium points in 2-dim. state-space

Consider finally the continuous-time system

$$\begin{aligned} \dot{x}_1 &= \mu x_1 - x_2 \pm x_1 \left(x_1^2 + x_2^2\right) &\quad (7.45) \\ \dot{x}_2 &= x_1 + \mu x_2 \pm x_2 \left(x_1^2 + x_2^2\right). &\quad (7.46) \end{aligned}$$

The origin $\overline{x} = (0, 0)$ is an equilibrium point for all $\mu \in \mathbb{R}$. The Jacobian matrix with respect to $x = (x_1, x_2)$ at this equilibrium point is

$$\frac{\partial F}{\partial x}((0, 0), \mu) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

and its eigenvalues are $\mu \pm j$. Therefore the equilibrium point $\overline{x} = (0, 0)$ is hyperbolic if and only $\mu \neq 0$, and by the implicit function theorem, in a neighborhood of $(\overline{x}_0, \mu_0) = ((0, 0), 0)$, there is only this equilibrium point.

It is convenient to transform system (7.45) - (7.46) into polar coordinates, by setting

$$\begin{aligned} r &= \left(x_1^2 + x_2^2\right)^{1/2} \\ \varphi &= \arctan\left(\frac{x_2}{x_1}\right). \end{aligned}$$

In these polar coordinates, the system (7.45) - (7.46) becomes

$$\begin{aligned} \dot{r} &= \mu r \pm r^3 &\quad (7.47) \\ \dot{\varphi} &= 1. &\quad (7.48) \end{aligned}$$

Equation (7.47) is nothing else but the normal form of the pitchfork bifurcation of an equilibrium point (7.40). Relating the 1-dim. flow of the pitchfork bifurcation to the 2-dim. flow of (7.45) - (7.46), we note that the equilibrium point $\overline{r} = 0$ corresponds to the equilibrium point $\overline{x} = (0, 0)$ and the equilibrium point $\overline{r}' = \sqrt{\pm\mu}$ (depending on the sign of $\mu$) corresponds to a periodic solution on a circular orbit of radius $\overline{r}'$ and period $2\pi$. The stability properties in 1-dim and 2-dim are the same, except that the periodic orbit is not asymptotically stable, but only stable, because of the indeterminate phase. The two types of qualitatively different bifurcations are represented in Figure 7.9.

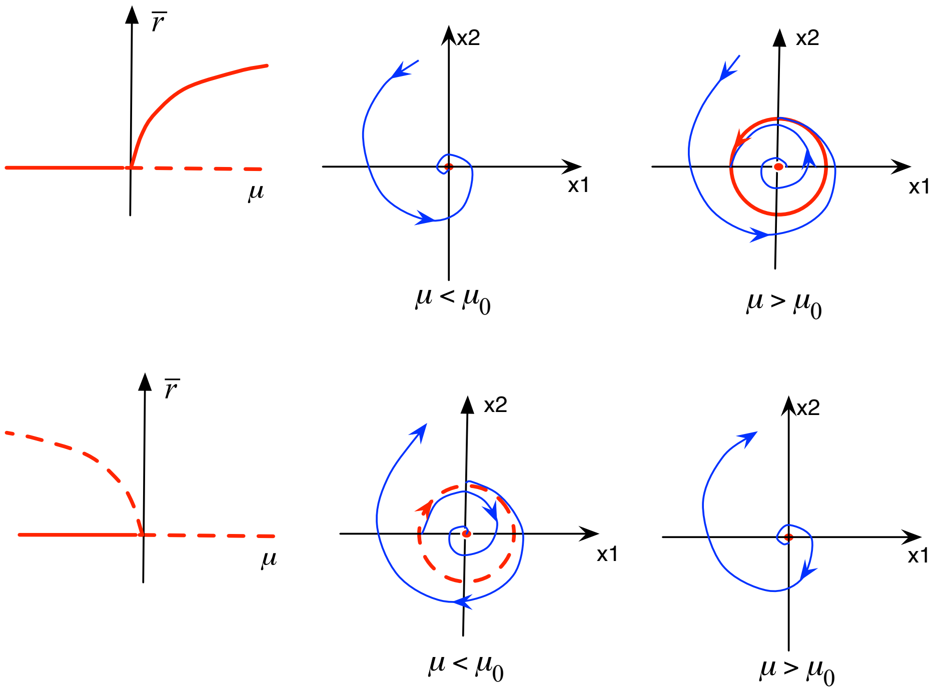


Figure 7.9: Andronov-Hopf bifurcation of a 2-dim. continuous-time system: supercritical (top) and subcritical (bottom), with $\overline{x}_0 = (0, 0)$. On the left, bifurcation diagram representing the family of equilibrium points in a neighborhood of $(\overline{x}_0, \mu_0)$. The solid line is the asymptotically stable equilibrium, the dotted line is the unstable equilibrium point(s). On the right, dynamics in the 1-dimensional state space for $\mu < \mu_0$ and $\mu > \mu_0$.

The following theorem puts the Andronov-Hopf bifurcation in a more general context than its normal form (7.45) - (7.46), for the equation (7.1). By the implicit function theorem, in a neighborhood of $(\overline{x}_0, \mu_0)$, there is the unique equilibrium point family $\overline{x}(\mu)$. We denote the eigenvalues of the Jacobian matrix $\frac{\partial F}{\partial x}(\overline{x}, \mu)$ in this neighborhood, which are complex conjugate for an Andronov-Hopf bifurcation, by $\lambda(\mu)$ and $\lambda^*(\mu)$.

**Theorem 7.6** (Andronov-Hopf Bifurcation in 2-dim systems)**.** *Let the continuous-time system given by*

$$\dot{x}(t) = F(x(t), \mu),$$

*and let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be a $C^4$-function. Let $\overline{x}_0 \in \mathbb{R}^2$ and $\mu_0 \in \mathbb{R}$ be such that $F(\overline{x}_0, \mu_0) = (0, 0)$ and that the Jacobian matrix $\partial F/\partial x(\overline{x}_0, \mu_0)$ has imaginary eigenvalues $\lambda_0 = j\omega_0$ and $\lambda_0^* = -j\omega_0$.*

*If*

$$\frac{d\Re(\lambda(\mu))}{d\mu}(\mu_0) \neq 0 \tag{7.49}$$

*and a complicated non-degeneracy condition is met, which we will not specify here (and which therefore can always be assumed to be satisfied in the exercises), then the system undergoes an Andronov-Hopf bifurcation at $(\overline{x}_0, \mu_0)$, that is, in a neighborhood of $(\overline{x}_0, \mu_0)$,*

*(i) for $\mu < \mu_0$, there is an asymptotically stable equilibrium point $\overline{x}(\mu)$, whereas for $\mu > \mu_0$ the equilibrium point $\overline{x}(\mu)$ becomes unstable, and in addition, there is a stable periodic solution, or vice-versa (this is called a* supercritical Andronov-Hopf bifurcation*) or for $\mu < \mu_0$, there is an asymptotically stable equilibrium point $\overline{x}(\mu)$ and an unstable periodic solution, whereas for $\mu > \mu_0$ there is only the equilibrium point $\overline{x}(\mu)$ and it is unstable, or vice-versa (this is called a* subcritical Andronov-Hopf bifurcation*);*

*(ii) by local, parameter-dependent continuous coordinate transformation, one can reduce the system to its normal form, which is*

$$\begin{aligned}\dot{x}_1 &= \mu x_1 - x_2 \pm x_1\left(x_1^2 + x_2^2\right)\\ \dot{x}_2 &= x_1 - \mu x_2 \pm x_2\left(x_1^2 + x_2^2\right)\end{aligned}$$

*(iii) the period of the periodic solution is a differentiable function $T(\mu)$ of $\mu$, with $T(\mu_0) = 2\pi/\omega_0$.*

## Example: Brusselator

The Brusselator is a model of chemical reaction proposed by Ilya Prigogine, which shows that some chemical reactions, such as the Belousov-Zhabotinsky reaction, can lead to oscillations. Its dimensionless equations with parameters $a$, $b$ read

$$\begin{aligned}\dot{x}_1 &= a - (b+1)x_1 + x_1^2 x_2\\ \dot{x}_2 &= b x_1 - x_1^2 x_2.\end{aligned}$$

The system has an equilibrium point at $\overline{x} = (a, b/a)$. Let us fix $a = 1$, and keep $b = \mu$ as a free parameter in $[1, 3]$. The eigenvalues of the Jacobian at $\overline{x} = (1, b)$ are

$$\lambda(b), \lambda^*(b) = \frac{1}{2}\left[b - 2 \pm \sqrt{b(b-4)}\right],$$

and are purely imaginary if $b = 2$. In addition,

$$\frac{d\Re(\lambda(b))}{db} = \frac{1}{2} \neq 0$$

and the Brusselator undergoes an Andronov-Hopf bifurcation at $(\overline{x}_0, b_0) = ((1, 2), 2)$. For $b < b_0 = 2$, $\overline{x}(b) = (1, b)$ is a stable focus, whereas for $b > b_0 = 2$, $\overline{x}(b)$ is an unstable focus, and a stable periodic solution appears, as shown in Figure 7.10. Hence the Andronov-Hopf bifurcation at $(\overline{x}_0, b_0) = ((1, 2), 2)$ is supercritical.

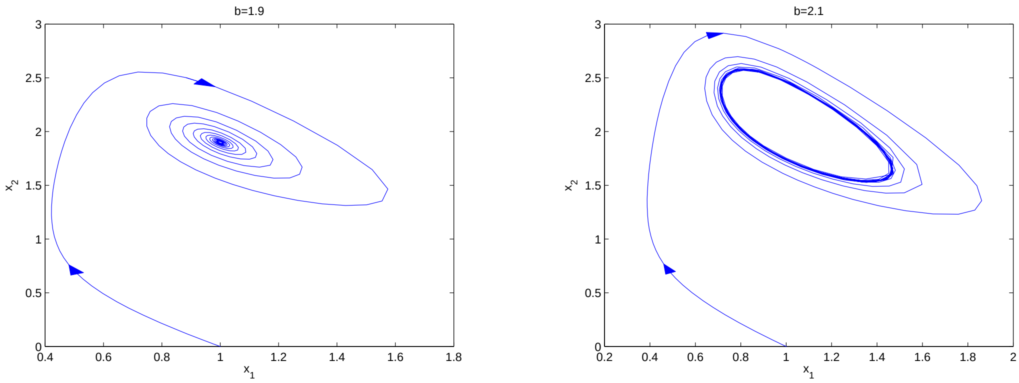


Figure 7.10: Phase portrait of a brusselator with $a = 1$, and $b = 1.9$ (left) or $b = 2.1$ (right).

### Counter-Example: Van der Pol Oscillator

It is not difficult to see that the Van der Pol oscillator (4.5) and (4.6) satisfies all conditions of Theorem 7.6 at $(\overline{x}_0, b_0) = ((0,0), 0)$ (Here the bifurcation parameter is $\lambda$). Nevertheless, it results from simulations that it is not an Andronov-Hopf bifurcation, because both for $\lambda < 0$ and for $\lambda > 0$, there is a periodic solution, unstable for $\lambda < 0$ and stable for $\lambda > 0$, as shown in Figure 4.2. Their size does not go to zero as $\lambda$ approaches $\lambda_0 = 0$. The transition at $\lambda_0 = 0$ is through a linear system with coexisting periodic solutions of all sizes (a center). It follows that the Van der Pol oscillator does not satisfy the last condition in Theorem 7.6, i.e. the condition that is not detailed in the theorem.

## 7.8 Other Bifurcation types

Bifurcations can of course occur in higher dimensional spaces as well, where the system is usually reduced to a bifurcating system in minimal state space dimension such as the one(s) described previously.

We have seen some of the main bifurcations of equilibrium points; one-parameter bifurcations of periodic solutions can be reduced to one-parameter bifurcations of equilibrium/fixed points as follows: a $T$-periodic solution of a discrete time system is a fixed point of the $T$th iteration of the corresponding map. A periodic solution of a continuous time system is a fixed point of its Poincar map.

Finally, one should mention that bifurcations can involve multiple parameters.