Dynamical System Theory For Engineers
Midterm Test

School I&C, Master Course

NAME and First name:

Your answers are to be written in the space provided just after each question, hence if a page is unstapled, please mark your name on it. There is a total of 8 pages. Your answers and justifications must be clear, precise and complete. The notation $\dot{x}$ stands for dx/dt.

Maximum: 20 points

Question 1 (3 points)

Consider a continuous-time dynamical system, with state space $\Omega = \mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$, whose state equation is

$$\dot{x} = F(x)$$

with

$$F(x) = \begin{cases} -x \ln x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0. \end{cases} \tag{1}$$

This system is known to admit one unique solution $x(t)$ for each initial state $x(0) \in \Omega$.

1. (1pt) List all the attractor(s) of this system, if any. Justify your answer.

2. (1pt) Let $\{\xi(t), t \in \mathbb{R}^+\}$ denote the solution of this system with initial condition $\xi(0) = 2$. What are its omega-limit set $\mathcal{S}_\omega(\xi)$ and its alpha-limit set $\mathcal{S}_\alpha(\xi)$?

3. (1pt) We have seen that a sufficient condition for a continuous-time dynamical system to admit exactly one solution is that $F(x)$ is continuous and locally Lipschitz with respect to x, i.e. that for any closed bounded set $X \subset \Omega$ there is some finite $k > 0$ such that $|F(x) - F(x')| \leq k\,|x - x'|$ for all $x, x' \in X$. Is $F(x)$ a locally Lipschitz continuous function for all $x \in \Omega$? Justify your answer. The system $\dot{x} = F(x)$ with $F(x)$ given by (1) is known to admit one unique solution $x(t)$ for each initial state $x(0) \in \Omega$. What can you conclude about the necessity for $F(x)$ to be locally Lipschitz for guaranteeing uniqueness of solutions?

Question 2 (3 points)

Consider the autonomous discrete-time linear system in $\mathbb{R}^2$ whose state equations are given by

$$\begin{aligned} x_1(t+1) &= \alpha x_1(t) + x_2(t) \\ x_2(t+1) &= x_1(t) + \alpha x_2(t), \end{aligned}$$

where $\alpha \in \mathbb{R}$ is a parameter. Characterize the stability of the system (i.e. asymptotically stable, stable, weakly unstable, strongly unstable) as a function of $\alpha \in \mathbb{R}$. Justify your answer.

Question 3 (6 points)

The state and output equations of a non-autonomous continuous-time linear system in $\mathbb{R}^2$ are

$$
\begin{aligned}
\dot{x}_1(t) &= x_1(t) - x_2(t) \\
\dot{x}_2(t) &= x_1(t) - x_2(t) + \beta u(t) \\
y(t) &= x_1(t) - x_2(t),
\end{aligned}
$$

where $\beta \in \mathbb{R}$ is a parameter.

1. (2pts) If $\beta = 0$, the system boils down to an autonomous system. Characterize its stability (i.e. asymptotically stable, stable, weakly unstable, strongly unstable). Justify your answer.

2. (2pts) Keeping again $\beta = 0$, compute the solution $(x_1(t), x_2(t))$ of this system for all $t \geq 0$ for the initial condition $(x_1(0), x_2(0)) = (2, 1)$.

3. If $\beta \neq 0$, the system is non autonomous. Is(are) there any value(s) $\beta \neq 0$ for which this system is B.I.B.O. stable? If so, determine all of them. If not, show that the system is never B.I.B.O. stable when $\beta \neq 0$.

Question 4 (8 points)

Consider an autonomous continuous-time nonlinear system in $\mathbb{R}^2$ given by

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 - 2x_2(2x_2^2 - 2x_1^2 + x_1^4).\end{aligned}$$

1. (5pts) Find all the equilibrium points of the system and characterize their stability (i.e., asymptotically stable, stable, unstable). Sketch, as precisely as possible, the phase portrait of the system in a small neighborhood around each of them. Justify your answer.

2. (3pts) Does this system have uniformly asymptotically bounded solutions? Justify your answer. Hint: a good Lyapunov function candidate would be $W(x_1, x_2) = x_2^k + (x_1^l - 2x_1^m + C^2)$ for some even integers k, l, m and where $C > 0$ is a constant that you pick to make $W(x_1, x_2) \geq 0$ for all $(x_1, x_2) \in \mathbb{R}^2$.