Solutions 3

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Solution 1. Power Spectrum Density
The process $Y[n]$ is obtained by filtering the wide-sense stationary process $X[n]$, i.e.

$$Y[n] = H(z)X[n],$$

with $H(z) = 1 + \beta z^{-1}$. Therefore,

$$S_Y(\omega) = |H(e^{j\omega})|^2 S_X(\omega).$$

The function $|H(e^{j\omega})|^2$ is given by

$$|H(e^{j\omega})|^2 = 1 + 2\beta\cos\omega + \beta^2.$$

The PSD of $X[n]$ is computed by taking the DTFT of $R_X[n]$, that gives

$$S_X(\omega) = \sum_{k=-\infty}^{\infty} R_X[k]e^{-j\omega k} = \sigma_X^2 \frac{1-\alpha^2}{1 - 2\alpha\cos\omega + \alpha^2}.$$

Hence, the PSD of $Y[n]$ is

$$S_Y(\omega) = \sigma_X^2(1-\alpha^2)\frac{1 + 2\beta\cos\omega + \beta^2}{1 - 2\alpha\cos\omega + \alpha^2}.$$

To have that $Y[n]$ is a white process, we should impose that the spectral density is a constant. This corresponds to setting $\beta = -\alpha$. The interpretation is the following. The process $X[n]$ is an AR process. In fact, it can be obtained by filtering a white noise $W[n]$, which has variance $\sigma_X^2(1-\alpha^2)$, with the synthesis filter

$$H_s(z) = \frac{1}{1 - \alpha z^{-1}},$$

which has a pole for $z = \alpha$. The filter $H(z)$ is an FIR filter (i.e. it has only zeros) and has exactly one zero at $z = -\beta$. We can imagine that the process $Y[n]$ is obtained by filtering $W[n]$ with the cascade of the filters $H_s(z)$ and $H(z)$. Therefore, to obtain a white noise at the output, we must have that the zero of $H(z)$ cancels the pole of $H_s(z)$, i.e. $-\beta = \alpha$.

Solution 2.
We will find a predictor of $X[n+2]$ using two different methods and see that they yield the same result. First, from the definition of X, we know that:

$$X[n+2] = X[n+1] - X[n] + 2X[n-1] + W[n+2] \tag{1}$$

(a) Projection theorem.

We want to find $Y[n-1]$ such that $\mathbb{E}\big((X[n+2]-Y[n-1])Y[n-k]\big)$ for $k=1,2,3$. We can write Y as:

$$Y[n-1]=\sum_{k=1}^{3} a_k X[n-k]$$

To find a projection, we need to solve the system of equations (for $k=1,2,3$), so that the difference between our predictor Y and process X is orthogonal "to the past":

$$\mathbb{E}\big((X[n+2]-a_1X[n-1]-a_2X[n-2]-a_3X[n-3])X[n-k]\big)=0$$

In order to calculate actual values of a_i, we have to first rewrite the system in terms of the auto-correlation:

$$R_X[2+k]-a_1R_X[k-1]-a_2R_X[k-2]-a_3R_X[k-3]=0. \tag{2}$$

On the other hand, for an AR process we have Yule-Walker equations:

$$R_X[m]+\sum_k p_kR_X[m-k]=\delta_m S_W^2$$

Where in our case $p_1=1$, $p_2=-1$ and $p_3=2$.

From these equations we find $R_X[m]$ for $m=0,1,\dots 5$. Those values can then be plugged into (2). Finally, (2) can be solved for a_i.

(b) Intuitive property.

We again start with (1). We need to expand terms dependent on n and $n+1$ and write them in terms of the past. We first expand $X[n+1]$:

$$\begin{aligned}X[n+2]&=X[n+1]-X[n]+2X[n-1]+W[n+2]\\&=(X[n]-X[n-1]+2X[n-2]+W[n+1])-X[n]+2X[n-1]+W[n+2]\\&=X[n-1]+2X[n-2]+W[n+1]+W[n+2].\end{aligned}$$

The first two terms of the equations are functions of the past, and the last two terms are innovations that are entirely independent of the past. Therefore, using intuitive property, we get that Y, the best linear predictor of X is:

$$Y[n-1]=X[n-1]+2X[n-2]$$

We can see that the second solution is much easier on paper.

Solution 3.

1) The recursion that allows to synthesize the process $X[n]$ from a white noise process $W[n]$ is:

$$X[n]=a_1X[n-1]+\ldots+a_MX[n-M]+W[n].$$

2) The correlation structure of X is.

$$R_X[n]=a_1R_X[n-1]+\ldots+a_MR_X[n-M]+\delta[n]\sigma_W^2.$$

3) According to Yule-Walker equations, compute the mean square error $\|\epsilon_m\|_2^2$ of the linear prediction for increasing orders m. This error will strictly decrease until order $M+1$, then it will be a constant. This is how you compute the order M of the ARMA process. Then you can solve the Yule-Walker equations of order$M$ to get the parameters of the AR model. The mean square prediction error of the predictor of order M corresponds to the variance of the input noise W.

4)

$$H(z) = \frac{1}{1 - a_1 z^{-1} - \ldots - a_M z^{-M}}.$$

Then $X(z) = H(z)W(z)$. Using the fundamental filtering formula, we obtain

$$S_X(\omega) = |H(e^{j\omega})|^2 \sigma_W^2.$$