Solutions 5

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Solution 1.
The AF method proceeds as follows:

- Let
$$x[n] = \sum_{k=1}^{K} \alpha_k e^{j w_k n}. \text{ (Notice the sign flipped w.r.t the lecture notes.)} \tag{1}$$
Find a $K+1$-tap discrete filter h such that $(h \star x)[n] = 0$. This is achieved by solving the Toeplitz system
$$\begin{bmatrix} x[n] & \cdots & x[n-K] \\ \vdots & \ddots & \vdots \\ x[n+K] & \cdots & x[n] \end{bmatrix} \begin{bmatrix} h[0] \\ \vdots \\ h[K] \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2}$$

- Factorize $H(z)$ into a product of K order-1 polynomials, the zeros of which correspond to the signal parameters $\{\omega_k\}_{k=1,\ldots,K}$:
$$H(z) = \prod_{k=1}^{K} \left(1 - e^{j\omega_k} z^{-1}\right). \tag{3}$$
This non-linear step is best done numerically.

- Find $\{\alpha_k\}_{k=1,\ldots,K}$ by solving the linear system
$$\begin{bmatrix} x[n] \\ \vdots \\ x[n+K-1] \end{bmatrix} = \begin{bmatrix} e^{j\omega_1 n} & \cdots & e^{j\omega_K n} \\ \vdots & \ddots & \vdots \\ e^{j\omega_1 (n+K-1)} & \cdots & e^{j\omega_K (n+K-1)} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_K \end{bmatrix}. \tag{4}$$

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(a) $\{\alpha_k^d, \omega_k^d\}_{k=1,\ldots,K}$ estimated through straightforward application of (2), (3) and (4).

(b) Let $\tilde{x}[n] = x(nT)$, where $T < \pi/\Omega$. Then
$$\tilde{X}\left(e^{j\omega}\right) = \frac{1}{T} X\left(\frac{\omega}{T}\right) = \sum_{k=1}^{K} \underbrace{\frac{2\pi}{T}\alpha_k^c}_{\alpha_k^d} \underbrace{\delta\left(\omega - \frac{\omega_k^c}{T}\right)}_{\delta(\omega - \omega_k^d)}.$$
Since $\tilde{X}\left(e^{j\omega}\right)$ is K-sparse, $\{\alpha_k^d, \omega_k^d\}_{k=1,\ldots,K}$ can be estimated via (2), (3) and (4). $\{\alpha_k^c, \omega_k^c\}_{k=1,\ldots,K}$ are then found by suitably rescaling $\{\alpha_k^d, \omega_k^d\}_{k=1,\ldots,K}$.

(c) Since $x(t)$ is T_p-periodic, its Fourier Series coefficients are given by

$$x^{FS}[n] = \frac{1}{T_p}\int_0^{T_p}\sum_{k=1}^{K}\alpha_k^c h(t - t_k^c)\exp\left(-j\frac{2\pi}{T_p}nt\right) = \sum_{k=1}^{K}\alpha_k^c\frac{1}{T_p}\int_0^{T_p} h(t - t_k^c)\exp\left(-j\frac{2\pi}{T_p}nt\right)$$

$$= \sum_{k=1}^{K}\alpha_k^c h^{FS}[n]\exp\left(-j\frac{2\pi}{T_p}t_k^c n\right).$$

$$\Longrightarrow \tilde{x}[n] = \frac{x^{FS}[n]}{h^{FS}[n]} = \sum_{k=1}^{K}\underbrace{\alpha_k^c}_{\alpha_k^d}\underbrace{\exp\left(-j\frac{2\pi}{T_p}t_k^c n\right)}_{\exp\left(j\omega_k^d n\right)}.$$

Since $\tilde{X}\left(e^{j\omega}\right)$ is K-sparse, $\{\alpha_k^d, \omega_k^d\}_{k=1,\ldots,K}$ can be estimated via (2), (3) and (4). $\{\alpha_k^c, \omega_k^c\}_{k=1,\ldots,K}$ are then found by suitably rescaling $\{\alpha_k^d, \omega_k^d\}_{k=1,\ldots,K}$.

Note that the AF method only works if one is able to compute h^{FS} and x^{FS} from samples of $h(t)$ and $x(t)$ explicitly.[1] This is possible via the *Fast Fourier Series* algorithm (FFS). The sampling period should be $T = T_p/N_{FS}$, where N_{FS} is the bandwidth of the T_p-periodic function $\sum_q h(t - qT_p)$.

(d) Let $h(t)$ be a known bandlimited pre-filter to an ADC such that

$$\tilde{x}(t) = (h \star x)(t) = \sum_q\sum_{k=1}^{K}\alpha_k^c h(t - qT_p - t_k^c).$$

$\tilde{x}(t)$ is T_p-periodic and bandlimited, hence we are back in the setup of (c).

[1] h^{FS} could also be computed analytically since $h(t)$ is known.