STATISTICAL SIGNAL AND DATA PROCESSING THROUGH APPLICATIONS
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Solutions 4

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Solution 1.

(a) The AR process is of order 3.

(b) According to the Yule-Walker equations,

$$\begin{bmatrix} R_X[0] & R_X[1] \\ R_X[1] & R_X[0] \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} R_X[1] \\ R_X[2] \end{bmatrix}.$$

Knowing that,

$$\begin{aligned} R_X[1] &= \mathbb{E}[X[n]X[n-1]] \\ &= 0.3\mathbb{E}[X[n-1]X[n-1]] - 0.4\mathbb{E}[X[n-2]X[n-1]] + 0.5\mathbb{E}[X[n-3]X[n-1]] + \mathbb{E}[W[n]X[n-1]] \\ &= 0.3R_X[0] - 0.4R_X[1] + 0.5R_X[2], \end{aligned}$$

and

$$\begin{aligned} R_X[2] &= \mathbb{E}[X[n]X[n-2]] \\ &= 0.3\mathbb{E}[X[n-1]X[n-2]] - 0.4\mathbb{E}[X[n-2]X[n-2]] + 0.5\mathbb{E}[X[n-3]X[n-2]] + \mathbb{E}[W[n]X[n-2]] \\ &= 0.3R_X[1] - 0.4R_X[0] + 0.5R_X[1], \end{aligned}$$

one obtains

$$\begin{aligned} a &= \tfrac{2}{15} \simeq 0.133 \\ b &= -\tfrac{1}{3} \simeq -0.333. \end{aligned}$$

Solution 2. Spectral estimation

(a) The Fourier series coefficients of $x(t)$ are given by

$$\widehat{x}[m] = \frac{1}{T}\int_0^T \sum_{k=0}^{M-1} a_k \delta(t-t_k) e^{-j\frac{2\pi}{T}mt} dt = \frac{1}{T}\sum_{k=0}^{M-1} a_k e^{-j\frac{2\pi}{T}mt_k}$$

(b) For $M = 3$, we have

$$\widehat{x}[m] = \frac{1}{T}(a_0 e^{-j\frac{2\pi}{T}mt_0} + a_1 e^{-j\frac{2\pi}{T}mt_1} + a_2 e^{-j\frac{2\pi}{T}mt_2})$$

Using the annihilating filter method, we will choose a filter of length 4 being $[1, h_1, h_2, h_3]$. We have then $H(Z)X(m) = 0$. In matrix notation we get

$$\begin{bmatrix} \widehat{x}[3] & \widehat{x}[2] & \widehat{x}[1] & \widehat{x}[0] \\ \widehat{x}[4] & \widehat{x}[3] & \widehat{x}[2] & \widehat{x}[1] \\ \widehat{x}[5] & \widehat{x}[4] & \widehat{x}[3] & \widehat{x}[2] \end{bmatrix} \begin{bmatrix} 1 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We thus need at least 6 components of Fourier series coefficients to have a unique solution.

(c) As already discussed in class, the anhilating filter method is not robust to addition of noise. For small variance of the noise and large values of the a_k coefficients, the algorithm will still perform well, but as soon as the SNR becomes too small, the algorithm cannot be used anymore.

Solution 3. Spectral factorization and estimation

The power spectral density $S_X(\omega)$ can be transformed in the following way:

$$\begin{aligned} S_X(\omega) &= \frac{b}{(1 + a_1^2 - 2a_1 \cos\omega)(1 + a_2^2 - 2a_2 \cos\omega)} \\ &= \frac{b}{|1 - a_1 e^{-j\omega}|^2 |1 - a_2 e^{-j\omega}|^2} \\ &= \frac{b}{|(1 - a_1 e^{-j\omega})(1 - a_2 e^{-j\omega})|^2} . \end{aligned}$$

(a) The power spectral density of an AR process has the form

$$S(\omega) = \frac{1}{|P(e^{j\omega})|^2} \sigma_W^2$$

where $P(z)$ is the minimum phase whitening filter, and σ_W^2 is the noise variance. Since $|a_1| < 1$ and $|a_2| < 1$, the polynomial $P(z) = (1 - a_1 z^{-1})(1 - a_2 z^{-1})$ is strictly minimum phase, and since $b > 0$, we can see that the PSD $S_X(\omega)$ corresponds to an AR process $P(z)X[n] = W[n]$, whose whitening filter is given by

$$P(z) = 1 - (a_1 + a_2)z^{-1} + a_1 a_2 z^{-2} ,$$

with the noise $W[n]$ having the variance $\sigma_W^2 = b$.

(b) Making the substitutions $p_1 = a_1 + a_2$ and $p_2 = -a_1 a_2$, we can write $P(z) = 1 - p_1 z^{-1} - p_2 z^{-2}$. The parameters p_1, p_2 and b can be determined by solving the following Yule-Walker equations:

$$\begin{aligned} b + p_1 \hat{R}_X[1] + p_2 \hat{R}_X[2] &= \hat{R}_X[0] \\ p_1 \hat{R}_X[0] + p_2 \hat{R}_X[1] &= \hat{R}_X[1] \\ p_1 \hat{R}_X[1] + p_2 \hat{R}_X[0] &= \hat{R}_X[2] , \end{aligned}$$

where $\hat{R}_X[0]$, $\hat{R}_X[1]$ and $\hat{R}_X[1]$ are the empirical correlation estimates at lags 0, 1 and 2.

Once the parameters p_1, p_2 and b have been determined, the parameters a_1 and a_2 can be determined by solving the non-linear system

$$\begin{aligned} a_1 + a_2 &= p_1 \\ a_1 a_2 &= -p_2 , \end{aligned}$$

under the constraints that $|a_1| < 1$ and $|a_2| < 1$. Furthermore, the estimated power spectral density $S_X(\omega)$ has the form

$$S_X(\omega) = \frac{b}{|1 - p_1 e^{-j\omega} - p_2 e^{-2j\omega}|^2} \, .$$