Solutions 2

Solution 1. One system or more than one system?

(a) From the z-plane plot we see that the two zeros are $z_1 = j\alpha$ and $z_2 = -j\alpha$, where $0 < \alpha < 1$ (real). Consequently the z-transform has the form $\tilde{H}(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1}) = (1 - j\alpha z^{-1})(1 + j\alpha z^{-1}) = 1 + j\alpha z^{-1} - j\alpha z^{-1} + \alpha^2 z^{-2} = 1 + \alpha^2 z^{-2}$. The corresponding impulse response is therefore $\tilde{h}(0) = 1$, $\tilde{h}(1) = 0$, $\tilde{h}(2) = \alpha^2$, and $\tilde{h}(k) = 0$ for $k \geq 3$. Hence, $\tilde{h}(n) = h(n)$ and $\tilde{H}(z) = H(z)$.

(b) The z-transforms shows two zeros near the unit circle at normalized frequencies $f_1 = 0.25$ and $f_2 = -0.25$. Consequently the magnitude of the corresponding frequency response $|\tilde{H}(e^{j2\pi f})|$ should show a minimum of the frequency response at $f_1 = 0.25$ and a minimum at $f_2 = -0.25$. The plot of $|H(e^{j2\pi f})|$ (bottom) shows a maximum at $f_1 = 0.25$ (and for the symmetry of the spectrum, a maximum at $f_2 = -0.25$). Consequently, $|H(e^{j2\pi f})| \neq |H(e^{j2\pi f})|$ and the plot of the z-transform and the plot of the magnitude of the frequency response $|H(e^{j2\pi f})|$ do not correspond to the same system.

Solution 2. Hilbert Spaces in Probability.

This exercise may seem strange at a first glance, but it is actually a standard exercise of linear algebra. One should just replace the scalar product used in $\mathbb{R}^N$ with

$$\langle X, Y \rangle = \mathbb{E}[XY],$$

where X and Y are random variables defined on the same probability space. One could easily verify that this product is actually a valid scalar product. The scalar product always induces a norm, defined by

$$\|X\| = \sqrt{\langle X, X \rangle}.$$

With these definitions, the space H is actually a Hilbert space. (One could verify that all the properties valid for vector spaces hold for the set H and also that H is complete.)
The space H is generated by the random variables X_0, X_1, X_2 which represent a basis of the space. They are the vectors of the space and one can apply the usual vector operations on them.

(a) The subspace W is the subspace of H generated by the vectors (i.e. the random variables) X_0 and X_1. To determine an orthogonal basis, one can apply the Gram-Schmidt procedure:

$$\begin{aligned} Y_0 &= \frac{X_0}{\|X_0\|} \\ Y_1 &= \frac{X_1 - \langle X_1, Y_0 \rangle Y_0}{\|X_1 - \langle X_1, Y_0 \rangle Y_0\|}, \end{aligned}$$

and replace the scalar product and the norm with the definitions that we presented earlier. We obtain,

$$\begin{aligned} Y_0 &= \frac{X_0}{2\sqrt{2}} \\ Y_1 &= \frac{X_1 - \langle X_1, \frac{X_0}{2\sqrt{2}} \rangle \frac{X_0}{2\sqrt{2}}}{\|(X_1 - \langle X_1, \frac{X_0}{2\sqrt{2}} \rangle \frac{X_0}{2\sqrt{2}}\|} \\ &= -\frac{1}{2\sqrt{6}} X_0 + \frac{1}{\sqrt{6}} X_1. \end{aligned}$$

(b) To determine the best approximation of X_2 in W, say $\hat{X}_2$, we write it as a linear combination of X_0 and X_1 (or equivalently of Y_0 and Y_1),

$$\hat{X}_2 = b_0 X_0 + b_1 X_1.$$

The error of the approximation is given by

$$E = X_2 - \hat{X}_2$$

To apply the projection theorem we impose that the approximation error is orthogonal to W. This correspond to the two equations:

$$\begin{aligned} \langle E, X_0 \rangle &= 0 \\ \langle E, X_1 \rangle &= 0, \end{aligned}$$

which gives the linear system:

$$\begin{cases} \langle X_0, X_0 \rangle b_0 + \langle X_1, X_0 \rangle b_1 &= \langle X_2, X_0 \rangle \\ \langle X_0, X_1 \rangle b_0 + \langle X_1, X_1 \rangle b_1 &= \langle X_2, X_1 \rangle. \end{cases}$$

The solution of the system is

$$\begin{aligned} b_0 &= -\tfrac{1}{6} \\ b_1 &= \tfrac{7}{12}; \end{aligned}$$

therefore, $\hat{X}_2 = -X_0/6 + 7X_1/12$.

Solution 3. Links between definitions

(a) **True** If a WSS process is real valued, then it's (auto)correlation is symmetric. Since the Power Spectral Density is defined as the (Discrete Time) Fourier Transform of the correlation of the process:

$$S_X(\omega) = \sum_{k=-\infty}^{\infty} R_X[k] e^{-ik\omega},$$

we can use the property that the Fourier Transform of a symmetric signal is real valued.

Alternatively, we can use the intuition from the lecture, that the Power Spectral Density is the expected value of the square of the Fourier Transform of the signal, and thus it's real valued.

Additional remark, not in the scope of the class: if you want to define the Power Spectral Density for a non stationary signal, you have to do this locally, because the signal is going to change over time, and you can't rely on R_X. You can define the Power Spectral Density as exactly the expected value of the square of the Fourier Transform of the signal on some interval:

$$\begin{aligned} \hat{S}_{X,N}(\omega) &= \mathbb{E} \left| \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] e^{-jk\omega} \right|^2 \\ &= \frac{1}{N} \mathbb{E} \left(\sum_{k=0}^{N-1} \sum_{m=0}^{N-1} X[k] X^*[m] e^{-j\omega(k-m)} \right) \end{aligned}$$

Then, if you assume that the signal is actually stationary, you can simplify this expression as follows:

$$\begin{aligned}\hat{S}_{X,N}(\omega) &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}\mathbb{E}\left(X[k]X^*[m]e^{-j\omega(k-m)}\right)\\ &= \sum_{l=-N+1}^{N-1}\frac{N-l}{N}R_X[l]e^{-jl\omega} \xrightarrow[N\to\infty]{} \sum_{l=-\infty}^{\infty}R_X[l]e^{-jl\omega}\end{aligned}$$

And therefore in the limit we get our "standard" PSD:

$$\hat{S}_{X,N}(\omega) \xrightarrow[N\to\infty]{} S_X(\omega)$$

which formalises intuition that the Power Spectral Density is exactly the expected value of the square of the Fourier Transform.

(b) **False**. A SSS process implies that it's values are indeed identically distributed, but *not always* independent. A simple counterexample is the discrete process built as follows:

$$X[n] = X[0] = Y \text{ for every } n$$

where Y is a (non constant) random variable. The process is clearly SSS, the variables Y are identically distributed, but are dependent.

(c) **True** It follows from the properties of expected value. If X and Y are *independent*, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$, and therefore:

$$\mathbb{E}\big((X-\mathbb{E}(X))(Y-\mathbb{E}(Y)\big) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0,$$

so X and Y are uncorrelated.

Solution 4. A Simple AR Process

(a) The recursion formula

$$X[n+1] = aX[n] + W[n+1], \quad n \geq 0$$

yields

$$\begin{aligned}X[1] =& aX[0] + W[1]\\ X[2] =& aX[1] + W[2] = a^2X[0] + aW[1] + W[2]\\ X[3] =& aX[2] + W[3] = a^3X[0] + a^2W[1] + aW[2] + W[3]\\ &\vdots\\ X[n] =& a^nX[0] + a^{n-1}W[1] + a^{n-2}W[2] + \cdots + W[n]\\ =& a^nX[0] + \sum_{k=0}^{n-1}a^kW[n-k].\end{aligned}$$

Hence the mean of $X[n]$ is given by

$$\mathbb{E}\left[X[n]\right] = a^n \mathbb{E}\left[X[0]\right] + \sum_{k=0}^{n-1} a^k \mathbb{E}\left[W[n-k]\right] = 0$$

since both $\mathbb{E}\left[X[0]\right] = 0$ and $\mathbb{E}\left[W[j]\right] = 0, \quad \forall j \geq 0$.

The variance of $X[n]$ is given by

$$\begin{aligned}
\mathbb{E}\left[|X[n]|^2\right] &= \mathbb{E}\left[\left(a^n X[0] + \sum_{k=0}^{n-1} a^k W[n-k]\right)\left(a^n X[0] + \sum_{j=0}^{n-1} a^j W[n-j]\right)^*\right] \\
&= |a|^{2n}\mathbb{E}\left[|X[0]|^2\right] + \sum_{j=0}^{n-1} a^n a^{*j} \mathbb{E}\left[X[0]W^*[n-j]\right] \\
&+ \sum_{k=0}^{n-1} a^k a^{*n} \mathbb{E}\left[W[n-k]X^*[0]\right] \\
&+ \sum_{k,j=0}^{n-1} a^k a^{*j} \mathbb{E}\left[W[n-k]W^*[n-j]\right]
\end{aligned}$$

Recall that $\mathbb{E}\left[|X[0]|^2\right] = c^2$, the sequence of random variables $W[n]$ and $X[0]$ are independent and centered, thus $\mathbb{E}\left[X[0]W^*[n-j]\right] = \mathbb{E}\left[W[n-k]X^*[0]\right] = 0, \quad \forall 0 \leq j,k \leq (n-1)$ and $\mathbb{E}\left[W[n-k]W^*[n-j]\right] = \sigma^2\delta[j-k]$. Combining these observations, we have

$$\begin{aligned}
\mathbb{E}\left[|X[n]|^2\right] &= |a|^{2n}c^2 + \sigma^2 \sum_{k,j=0}^{n-1} a^k a^{*j}\delta[j-k] \\
&= |a|^{2n}c^2 + \sigma^2 \sum_{k=0}^{n-1} |a|^{2k} \\
&= |a|^{2n}c^2 + \sigma^2 \left(\frac{1-|a|^{2n}}{1-|a|^2}\right)
\end{aligned}$$

(b) If $c^2 = \frac{\sigma^2}{1-|a|^2}$, the variance of $X[n]$ is independent of n and is given by

$$\mathbb{E}\left[|X[n]|^2\right] = \frac{\sigma^2}{1-|a|^2}.$$

Following the same steps in part (a), one can show that

$$X[n+k] = a^k X[n] + a^{k-1}W[n+1] + a^{k-2}W[n+2] + \cdots + W[k+n]. \tag{1}$$

Thus

$$\begin{pmatrix} X[n] \\ X[n+1] \\ \vdots \\ X[n+k] \end{pmatrix} = A \begin{pmatrix} X[n] \\ W[n+1] \\ \vdots \\ W[n+k] \end{pmatrix}$$

The distribution of $(X[n], W[n+1], \cdots, W[n+k])$ is independent of $n \geq 0$, and therefore the distribution of $(X[n], \cdots, X[n+k])$ is independent of $n \geq 0$, therefore $\{X[n]\}_{n\geq 0}$ is strictly stationary.

(c) Recall that

$$X[n] - aX[n-1] = W[n],$$

thus $\langle X[n] - aX[n-1], u\rangle = \langle W[n], u\rangle = 0$ for all $u \in H(X, n-1)$ since $H(X, n-1) = H(W, n-1)$ (see Theorem 2.2 in class notes). Recall that, roughly speaking, $H(W, n-1)$ is composed of linear combinations of $W[n-1], W[n-2], \ldots$. Note also that $aX[n-1] \in H(X, n-1)$ (since it is a linear function of $X[n-1]$), hence by the projection theorem this is the best linear approximation (best in least square sense) for $X[n]$, thus $\widehat{X}[n|n-1] = aX[n-1]$.

(d) The whitening filter makes $\{X[n]\}_{n\geq 0}$ a white noise, here we have

$$X[n] - aX[n-1] = W[n],$$

so it is clear that $P(z) = 1 - az^{-1}$. The generating filter is given by

$$H^s(z) = \frac{1}{P(z)} = \frac{1}{(1 - az^{-1})} = \sum_{n\geq 0} a^n z^{-n}$$

and

$$X[n] = W[n] + aW[n-1] + a^2 W[n-2] + \cdots + a^k W[n-k] + \ldots$$

(e) Using (1) we obtain

$$\begin{aligned}
\mathbb{E}\left[X[n+k]X^*[n]\right] =& \mathbb{E}\left[\left(a^k X[n] + \sum_{j=0}^{k-1} a^j W[n+k-j]\right) X^*[n]\right] \\
=& a^k \mathbb{E}\left[|X[n]|^2\right] + \sum_{j=0}^{k-1} a^j \mathbb{E}\left[W[n+k-j]X^*[n]\right] \\
=& a^k \mathbb{E}\left[|X[n]|^2\right] \\
=& a^k \left(|a|^{2n} c^2 + \sigma^2 \left(\frac{1 - |a|^{2n}}{1 - |a|^2}\right)\right)
\end{aligned}$$

where the last equality follows from part (a) and $\mathbb{E}\left[W[n+k-j]X^*[n]\right] = 0$ since $W[n+k-j]$ and $X[n]$ are independent $\forall 0 \leq j \leq k-1$. Plugging $c^2 = \frac{\sigma^2}{1-|a|^2}$ yields

$$R_X[k] = a^k \frac{\sigma^2}{1 - |a|^2}.$$

Note that the above equality together with the fact that $\mathbb{E}\left[X[n]\right] = 0$ shows that the process $\{X[n]\}_{n\geq 0}$ is wide sense stationary with the special condition $c^2 = \frac{\sigma^2}{1-|a|^2}$. Since the process is wide sense stationary and Gaussian, it is strictly stationary. Recall that the statistics of a Gaussian process is completely determined by its first and second order properties.

(f) Again from (1), we have

$$X[n] = a^2 X[n-2] + aW[n-1] + W[n].$$

thus $\langle X[n] - a^2 X[n-2], u\rangle = \langle aW[n-1], u\rangle + \langle W[n], u\rangle = a\langle W[n-1], u\rangle + \langle W[n], u\rangle = 0$ for all $u \in H(X, n-2)$ since $H(X, n-2) = H(W, n-2)$ and the random variables

$W[n-2], W[n-3], \ldots$ are independent of $W[n-1]$ and $W[n]$. Note also that $a^2 X[n-2] \in H(X, n-2)$, hence by the projection theorem this is the best least square approximation for $X[n]$ knowing $X[n-2], X[n-3], \ldots$, thus $\widehat{X}[n|n-2] = a^2 X[n-2]$.