Solutions 7

Solution 1.

(a) The expression for the Wiener filter is

$$H(e^{j\omega}) = \frac{S_{XY}(\omega)}{S_Y(\omega)}$$

If $Y[n] = X[n] + W[n]$ and $X[n]$ and $W[n]$ are uncorrelated with $W[n]$ zero-mean, we have

$$\begin{aligned} R_{XY}[m] &= \mathbb{E}\left[X[n+m](X[n] + W[n])\right] = \mathbb{E}\left[X[n+m]X[n]\right] = R_X[m] \\ R_Y[m] &= \mathbb{E}\left[(X[n+m] + W[n+m])(X[n] + W[n])\right] = R_X[m] + R_W[m] \end{aligned}$$

hence,

$$\begin{aligned} S_{XY}(\omega) &= S_X(\omega) \\ S_Y(\omega) &= S_X(\omega) + S_W(\omega). \end{aligned}$$

The Wiener filter is thus

$$H(e^{j\omega}) = \frac{S_X(\omega)}{S_X(\omega) + S_W(\omega)}$$

(b)

$$\begin{aligned} H(e^{j\omega}) &= \frac{S_X(\omega)}{S_X(\omega) + S_W(\omega)} \\ &= \frac{a(\omega)}{a(\omega) + 1}. \end{aligned}$$

If $a(\omega_o) \gg 1$, $H(e^{j\omega}) \approx 1$. That is, the filter applies little or no attenuation to the noise-free frequency component. If $a(\omega_o) \approx 0$, $H(e^{j\omega}) \approx 0$. That is, the filter applies a high attenuation to the noisy frequency component. In conclusion, for additive noise, the Wiener filter attenuates each frequency component in proportion to an estimate of the signal to noise ratio.

Solution 2.

(a) By filtering the unit-variance white noise with a filter $H(e^{j\omega})$, we get an output signal $X[n]$ with power spectrum:

$$S_X(\omega) = |H(e^{j\omega})|^2.$$

The Wiener filter that estimates $X[n]$ from $Y[n]$:

$$\begin{aligned} Q(e^{j\omega}) &= \frac{S_X(\omega)}{S_X(\omega) + S_V(\omega)} \\ &= \frac{|H(e^{j\omega})|^2}{|H(e^{j\omega})|^2 + (1/2)}. \end{aligned}$$

This is sufficient solution, but if we want to get exact formula, we can start with calculating the square norm of $H(e^{j\omega})$:

$$|H(e^{j\omega})|^2 = \frac{1 + \frac{3}{2}\cos(\omega) + \frac{9}{16}}{1 + \cos(\omega) + \frac{1}{4}}$$

Let $N(e^{j\omega})$ be the numerator of the above expression, and $D(e^{j\omega})$ be the denominator of the expression.

$$\begin{aligned} Q(e^{j\omega}) &= \frac{2N(e^{j\omega})}{2N(e^{j\omega}) + M(e^{j\omega})} \\ &= \frac{3\frac{1}{8} + 3\cos(\omega)}{5\frac{3}{8} + 4\cos(\omega)} \end{aligned}$$

(b) In this case, the signal $V[n]$ is not independent of the signal $X[n]$. Therefore, we can not apply the formula derived in part (a). We have to calculate the Wiener filter expression using the general formula, that is:

$$Q(e^{j\omega}) = \frac{S_{XY}(\omega)}{S_Y(\omega)}$$

We need to calculate S_{XY} and S_Y. We will first calculate S_Y in terms of S_{XY} and see how it simplifies the calculations. Let's calculate R_Y in terms of R_{XY}:

$$\begin{aligned} R_Y[m] &= \mathbb{E}\left[Y[n](X[n] + V[n])\right] \\ &= \mathbb{E}\left[Y[n]X[n-m]\right] + \mathbb{E}\left[\frac{1}{3}Y[n]X[n-m-1]\right] - \mathbb{E}\left[\frac{1}{9}Y[n]X[n-m-2]\right] \\ &= R_{XY}[m] + \frac{1}{3}R_{XY}[m+1] - \frac{1}{9}R_{XY}[m+2]. \end{aligned}$$

After taking Fourier transform we get S_Y:

$$S_Y(\omega) = S_{XY}(\omega)\left(1 + \frac{1}{3}e^{j\omega} - \frac{1}{9}e^{2j\omega}\right)$$

And the Wiener filter is given by:

$$Q(e^{j\omega}) = \frac{S_{XY}}{S_Y} = \frac{1}{1 + \frac{1}{3}e^{-j\omega} - \frac{1}{9}e^{-2j\omega}}.$$

Solution 3.

(a)

$$
\begin{aligned}
\|\mathbf{M}\pi\|_1 &= \sum_{i=1}^{n}|(\mathbf{M}\pi)_i| = \sum_{i=1}^{n}|(\sum_{j=1}^{n}\pi_j \mathbf{m}_j)_i| = \sum_{i=1}^{n}|\sum_{j=1}^{n}\pi_j m_{ij}| \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\pi_j m_{ij} = \sum_{j=1}^{n}\pi_j \sum_{i=1}^{n} m_{ij} = \sum_{j=1}^{n}\pi_j \cdot 1 = \sum_{j=1}^{n}|\pi_j| \\
&= \|\pi\|_1 \,.
\end{aligned}
$$

(b) No, it does not. A counterexample are the following matrix and vector

$$
\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \pi_{\mathbf{0}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1}
$$

It is easy to verify that the sequence does not converge.

Solution 4. Markov chain
Solution is available in Jupyter Notebook