Solutions 9

Solution 1. CORRELATING AND DECORRELATING SIGNALS

(a) The set of random variables $\mathbf{Y}$ is determined as

$$Y_i = \sum_{k=0}^{N-1} \alpha_{i,k} \cdot X_k.$$

The correlation function is given by

$$\begin{aligned} R_{i,j} &= \mathbb{E}[Y_i \cdot Y_j] = \mathbb{E}[\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \alpha_{i,k}\alpha_{j,l} X_k X_l] \\ &= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \alpha_{i,k}\alpha_{j,l} \mathbb{E}[X_k X_l]. \end{aligned}$$

Since the random variables X_i are normalized and independent, the correlation simplifies to

$$R_{i,j} = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \alpha_{i,k}\alpha_{j,l} \delta[k-l] = \sum_{k=0}^{N-1} \alpha_{i,k}\alpha_{j,k}.$$

(b) From (a) we can see that the correlation matrix $\mathbf{R}_Y$ is defined as $\mathbf{R}_Y = \mathbf{A} \cdot \mathbf{A}^T$. If we apply the det operator, we obtain

$$\det(\mathbf{R}_Y) = \det(\mathbf{A}) \cdot \det\left(\mathbf{A}^T\right) = (\det(\mathbf{A}))^2. \tag{1}$$

The correlation matrix $\mathbf{R}_y$ can also be expressed in terms of its eigenvalues and eigenvectors as

$$\mathbf{R}_Y = \mathbf{V}_Y \cdot \mathbf{\Lambda}_Y \cdot \mathbf{V}_Y^T,$$

where $\mathbf{V}_y$ is a matrix containing the eigenvectors as columns and $\mathbf{\Lambda}_y$ is a diagonal matrix with the eigenvalues along the diagonal.

Similarly, we can write

$$\det(\mathbf{R}_Y) = \det(\mathbf{V}_Y) \cdot \det(\mathbf{\Lambda}_Y) \cdot \det\left(\mathbf{V}_Y^T\right) = \prod_{i=0}^{N-1} \lambda_i \tag{2}$$

because $\mathbf{\Lambda}_Y$ is diagonal and $\det(\mathbf{V}_Y) = 1$. Therefore from (1) and (2), we have

$$\det(\mathbf{R}_Y) = (\det(\mathbf{A}))^2 = \prod_{i=0}^{N-1} \lambda_i.$$

It follows that

$$\det(\mathbf{A}) = \prod_{i=0}^{N-1} \lambda_i^{1/2}.$$

(c) The KLT matrix $\mathbf{T}$ is given by $\mathbf{T} = \mathbf{V}_Y^T$ because it contains the eigenvectors of the correlation matrix as the rows. Therefore, we can write

$$\mathbf{Z}[n] = \mathbf{V}_Y^T \cdot \mathbf{Y}[n].$$

The correlation $\mathbf{R}_Z$ is given by $\mathbf{R}_Z = \mathbb{E}\left[\mathbf{Z}[n] \cdot \mathbf{Z}^T[n]\right]$. It follows

$$\mathbf{R}_Z = \mathbb{E}\left[\mathbf{V}_Y^T \cdot \mathbf{Y}[n] \cdot \mathbf{Y}^T[n] \cdot \mathbf{V}_Y\right] = \mathbf{V}_Y^T \cdot \mathbf{R}_Y \cdot \mathbf{V}_Y.$$

From (b) we know that $\mathbf{R}_Y = \mathbf{V}_Y \cdot \mathbf{\Lambda}_Y \cdot \mathbf{V}_Y^T$. Therefore

$$\mathbf{R}_Z = \mathbf{V}_Y^T \cdot \mathbf{V}_Y \cdot \mathbf{\Lambda}_Y \cdot \mathbf{V}_Y^T \cdot \mathbf{V}_Y = \mathbf{\Lambda}_Y,$$

because $\mathbf{V}_Y^T \cdot \mathbf{V}_Y = \mathbf{I}$. The correlation matrix $\mathbf{R}_Z$ is diagonal and, thus, the variables $Z_i[n]$, for $i = 0, 1, \dots, N-1$, are uncorrelated. If the random variables are Gaussian, uncorrelation is equivalent to independence.

Solution 2. KLT of circulant correlation matrices

(a) The KLT matrix T is given by the eigenvectors of R_x:

$$T = \begin{bmatrix} -1/2 & 1/2 & -1/2 & 1/2 \\ 0 & -\sqrt(2)/2 & 0 & \sqrt(2)/2 \\ \sqrt(2)/2 & 0 & -\sqrt(2)/2 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}$$

To show that this is indeed a KLT matrix we compute TR_xT^T:

$$TR_xT^T = \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

which is a diagonal matrix.

(b) The DFT matrix $S_N[k,n] = W_N^{-kn}$ of size $N = 4$ is given by:

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}.$$

If we compute now $S_N^* R_x S_N$ we obtain:

$$S_N^* R_x S_N = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 3.2 & 0 & 0 \\ 0 & 0 & 1.6 & 0 \\ 0 & 0 & 0 & 3.2 \end{bmatrix},$$

which is also a diagonal matrix.

(c) Both transforms T and S_N give a diagonal correlation matrix and can be used as a decorrelation transform. However, the DFT matrix is constant for a given N and much easier to compute that the KLT matrix. However, the DFT matrix does not always produce the same results of the KLT. This exercise is a particular case where X is periodic and R_x is a circulant matrix. The reason is that the DFT matrix diagonalizes ANY circulant matrix. Therefore, if R_x is a circulant matrix, the DFT matrix is preferable as a decorrelation transform.

Solution 3. Automatic Classification of Sound Waves

The problem here is to denoise $y[n]$ which is a realization of a 4-state Markov chain $X[n]$ plus Gaussian noise $W[n]$, the latter being centered and having variance σ_W^2. Since $X[n]$ is discrete-valued, the only denoising approach is to use a mixture model (Markovian). We suppose the 4 states of the Markov chain $\{\alpha, \beta, \gamma, \delta\}$ are coded into 4 numerical values $\{m_1, m_2, m_3, m_4\}$.

- *Mixture Model*

  The distribution of $Y[n]$ can be written as a Markovian mixture model. Calling $\boldsymbol{y} = [y[1], \ldots, y[1000]]$ the realization of $\boldsymbol{Y} = [Y[1], \ldots, Y[1000]]$, and $\boldsymbol{X} = [X[1], \ldots, X[1000]]$ the corresponding underlying Markov process, we have

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = \sum_{\boldsymbol{x} \in \mathcal{X}} \prod_{n=1}^{1000} f_{x[n]}(y[n]) \mathbb{P}\left(\boldsymbol{X} = \boldsymbol{x}\right) = \sum_{\boldsymbol{x} \in \mathcal{X}} \prod_{n=1}^{1000} f_{x[n]}(y[n]) \pi_{x[1]} p_{x[1]x[2]} \cdots p_{x[999]x[1000]}$$

  where:

  – $\mathcal{X}$ represents the set of all the possible value combinations of $\boldsymbol{x} = [x[1], \ldots, x[1000]]$ where each $x[n]$ takes value in a set of four possible values $\{m_1, m_2, m_3, m_4\}$, $n = 1, \ldots, 1000$;

  – $f_{x[n]}(y[n]) = \mathcal{G}_{x[n], \sigma_W^2}(y[n])$ is a Gaussian distribution with mean given by the value $x[n]$ and variance σ_W^2, $n = 1, \ldots, 1000$;

  – $\pi_{x[1]}$, is the initial probability, $x[1] \in m_1, m_2, m_3, m_4$, and $p_{x[n]x[n+1]}$, $x[n], x[n+1] \in \{m_1, m_2, m_3, m_4\}$, $n = 1, \ldots, 999$, are the transition probabilities.

  Notice that the model parameters are

  – The means of the 4 Gaussian distributions m_i, $i = 1, \ldots, 4$;

  – The common variance of the 4 Gaussian distributions σ_W^2;

  – The initial probabilities π_{m_i}, that we can also write as π_i, $i = 1, \ldots, 4$;

  – The transition probabilities $\pi_{m_j m_i}$ that we can also write as π_{ji}, $j, i = 1, \ldots, 4$;

- *Parameter Estimation (MLE)*

  Applying the (modified) EM algorithm, to the likelihood function

$$h(\boldsymbol{y}|\theta = \{\sigma_W^2, m_i, \pi_i, p_{ij}; i, j = 1, \ldots, 4\}) = \sum_{\boldsymbol{x} \in \mathcal{X}} \prod_{n=1}^{1000} f_{x[n]}(y[n]) \pi_{x[1]} p_{x[1]x[2]} \cdots p_{x[999]x[1000]},$$

  enables the estimation of the parameters of the model $\widehat{\theta}$ (via the maximum likelihood technique).

  In particular, we obtain the 4 numerical values $\widehat{m}_i$, $i = 1, \ldots, 4$, of the Markov chain.

- *Denoising of the Markov Chain (MAP)*
  Once the parameters of the mixture model are known, we can estimate the denoised values of the Markov chain $\widehat{\boldsymbol{x}} = [\widehat{x}[1], \ldots, \widehat{x}[1000]]$ by maximizing the a-posteriori distribution

$$\widehat{\boldsymbol{x}} = \arg\max_{\boldsymbol{x}} P(\boldsymbol{X} = \boldsymbol{x} \mid \boldsymbol{y}) \, ,$$

  where

$$P(\boldsymbol{X} = \boldsymbol{x} \mid \boldsymbol{y}) \propto f_{\boldsymbol{Y}}(\boldsymbol{y} \mid \boldsymbol{X} = \boldsymbol{x}) \mathbb{P} \left(\boldsymbol{X} = \boldsymbol{x} \right) \, ,$$

  under the constraint

$$\sum_{n=1}^{1000} \prod_{i=1}^{4} (x[n] - \widehat{m}_i) = 0.$$