Statistical Signal & Data Processing - COM500
Midterm Exam

April 17 2025, Duration 1h

Name : Family Name : Seat N^o :

Read Me First!

You are allowed to use:

- The given cheatsheet summarizing the most important formulas;
- A pocket calculator.

You are definitively not allowed to use:

- Any kind of support not mentioned above;
- Your neighbor; Any kind of communication systems (smartphones etc.) or laptops;
- Printed material; Text and Solutions of exercises/problems; Lecture notes or slides.

Write solutions on separate sheets, *i.e.* no more than one solution per paper sheet.

Return your sheets ordered according to problem (solution) numbering.

All the best for your exam!!

Warmup Exercise

This is a warm up problem .. do not spend too much time on it. Please provide justified, rigorous, and simple answers. If needed, you can add assumptions to the problem setup.

Exercise 1. AR PROCESS (6 PTS)

Consider an AR process given by the following equation

$$X[n] = X[n-1] - 0.25X[n-2] + W[n]\,,$$

where $W[n]$ is a Gaussian white noise with variance $\sigma^2 = 1$.

1) Compute the correlation.
2) Compute the power spectral density.
3) Draw on the z-plane the poles and zeros of the synthesis filter.

Main Problem

Here comes the core part of the exam .. take time to read the introduction and each problem statement. Please provide justified, rigorous, and simple answers. Remember that you are not simply asked to describe statistical signal processing tools, but you are rather asked to describe how to apply such tools to the given problem. If needed, you can add assumptions to the problem setup.

Exercise 2. RIPPLES AND FAST RIPPLES IN EPILEPTIC EEG (26 PTS)

Electroencephalography (EEG) is a method to record an electrogram of the neural (electrical) activity of the brain. It is typically non-invasive, with the EEG electrodes placed along the scalp.

During both human and experimental epilepsy, EEG measurements have shown two types of very particular signals: One called ripples (R), with oscillation frequencies in the range of 80–250 Hz, and the other fast ripples (FR), with oscillation frequencies in the range of 250–600 Hz. In epilepsy analysis frameworks, EEG signals have a sampling frequency of $f_s = 2$ kHz.

The figure below shows a time frequency analysis (spectrogram) of the simulated neural activity of an acute phase of epilepsy. Think of a spectrogram as a sequence of periodograms (power spectral densities), computed using a sequence of samples centered at different instant of times.

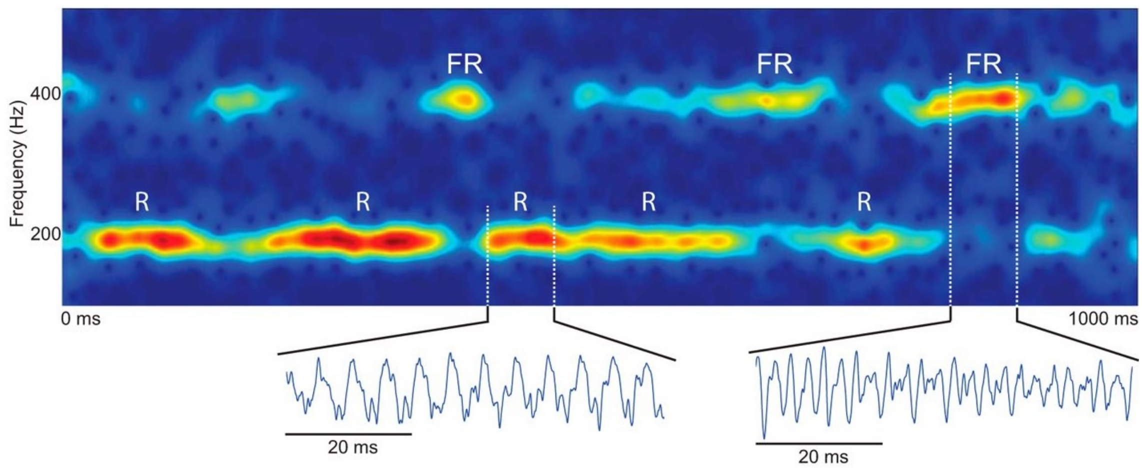


Fig. 1: Time frequency analysis of the simulated neural activity of an acute phase of epilepsy (Fink, C. G. and Gliske, S. and Catoni, N. and Stacey, W. C., "Network Mechanisms Generating Abnormal and Normal Hippocampal High-Frequency Oscillations: A Computational Analysis", eNeuro Vol. 2, Issue 3 May/June 2015).

In particular, we can observe a high energies around the frequency 200 Hz (ripples) and around the frequency 400 Hz (fast ripple).

We shall call $x[n]$ the signal of the neural activity.

1) Propose a simple w.s.s. model for the signal $x[n]$. For the sake of simplicity, you can assume that the EEG signal is the sum of a periodic component at frequency 200 Hz, representing the ripples, a periodic component 400 Hz, representing the fast ripples, and a white noise $W[n]$.

2) Prove that the proposed model is indeed w.s.s..

We have measured $N = 1000$ samples of $x[n]$. The time frequency plot of Fig. 1 shows two energy peaks, one at 200 Hz (ripples) and the other at 400 Hz (fast ripple).

3) Given the number of samples N, is it possible that there are more energy peaks, *i.e.*, more components around the frequencies 200 Hz and 400 Hz? If so, give that intervals at which these components might be.

We now drop the simple assumption that the signal can be modeled as two (or more) periodic components at precise frequencies, plus noise. We rather take into account that we actually have a distribution of energy around 200 Hz and 400 Hz. We would like to estimate the shape of such a distribution of energies, *i.e.*, the shape of the power spectral density, for a given instant (interval) of time.

4) Propose a parametric method to estimate the shape of the power spectral density. Precisely describe such method: You are given the 1000 samples, sampled at $f_s = 2\,\text{kHz}$, and you are asked to detail each step as if you have to implement the method in a computer. Precisely indicate the input and output of each step.

6) Write the expression (a sum with the right indexes) of the energy contained in the interval $[175, 225]$ Hz (energy of the ripples) and the expression (a sum with the right indexes) of the energy contained in the interval $[375, 425]$ Hz (energy of the fast ripples).

Stochastic Processes

w.s.s.

$$\mathrm{E}[X[n]]=\text{const.}\,,\mathrm{Var}(X[n])=\text{const.}<\infty$$

$$\mathrm{E}[X[k]X^*[l]]=R(k-l)\,,\forall k,l\in\mathbb{Z}\,,$$

PSD: w.s.s. + $R_X(k)\in\ell_1$ (summable)

$$S_X(\omega)=\sum_{k=-\infty}^{\infty}R_X(k)e^{-i\omega k}\,.$$

Fundamental Filtering Formula

$X[n]$ w.s.s., $R_X(k)\in\ell_1$, and $h_k\in\ell_1$, then

$$Y[n]=\sum_{k=-\infty}^{\infty}h_{n-k}X[k]\,,\text{ is w.s.s. with}$$

$$\mathrm{E}[Y]=\mathrm{E}[X]\sum_{k=-\infty}^{\infty}h_k\,,\quad R_Y(k)\in\ell_1$$

$$S_Y(\omega)=|H(\omega)|^2S_X(\omega)\,,H(\omega)=\text{DTFT of }h_k$$

Markov Chain

$\{X[n]\}_{n\in\mathbb{Z}}$ (considered stationary) with discrete values in $\mathcal{D}$, |

$$\mathrm{P}\big(X[n]=i_n\ \mid\ X[n-1]=i_{n-1},X[n-2]=i_{n-2},\ldots\big)$$
$$=\mathrm{P}\big(X[n]=i_n\ \mid\ X[n-1]=i_{n-1}\big)\,,\forall i_n,i_{n-1},\ldots\in\mathcal{D}$$

Hidden Markov Chain

$\{X[n]\}_{n\in\mathbb{Z}}$ Markov chain, $\{W[n]\}_{n\in\mathbb{Z}}$ Gaussian white noise

$$Y[n]=X[n]+W[n]\,.$$

Bayes' Rule

A and B with discrete values in $\mathcal{D}$,

$$\mathrm{P}(A=k\ \mid\ B=l)=\frac{\mathrm{P}(A=k\,,\ B=l)}{\mathrm{P}(B=l)}\,,\quad k,l\in\mathcal{D}\,.$$

AR Process

A w.s.s. process $X[n]$, with values in $\mathbb{R}$, |

$$\sum_{k=0}^{M}p_kX[n-k]=W[n]\,,\quad n\in\mathbb{Z}\,,$$

$W[n]$, is a zero mean Gaussian white noise

$p_k\,,k=0,\ldots,M$ bounded coefficients (real or complex). We assume $p_0=1$.

Filtering Interpretation (z^{-1} delay operator)

$$P(z)X[n]=W[n]$$

Canonical form: $P(z)$ strict. min. phase, $p_0=1$.

Correlation:

$$R_X[m]+\sum_{k=1}^{M-1}p_kR_X[m-k]=\delta_m\sigma_W^2\,,\quad m\geq 0\,.$$

PSD (fundamental filtering formula):

$$S_X(\omega)\big|P\big(e^{j\omega}\big)\big|^2=\sigma_W^2\,.$$

Harmonic Processes

$$X[n]=\sum_{k=1}^{K}\alpha_ke^{j(\omega_kn+\Theta_k)}\,,n\in\mathbb{N}\,,$$

Θ_k i.i.d. uniformly distributed over $[0,2\pi]$.

$$R_X[l]=\sum_{k=1}^{K}|\alpha_k|^2e^{j\omega_kl}\,,\ S_X(\omega)=\sum_{k=1}^{K}|\alpha_k|^2\delta(\omega-\omega_k)\,.$$

Poisson Random Process $N((0,t])$

$N((0,t])$ obeys the Poisson distribution $\mathrm{P}(N=k)=\frac{(a\lambda)^ke^{-a\lambda}}{k!}$, ($\lambda$ is the rate), and given two disjoint intervals $(t_1,t_2]$ and $(t_3,t_4]$, $N((t_1,t_2])$ is independent of $N((t_3,t_4])$.

Inter-arrival time $S_n=T_n-T_{n-1}$. i.i.d. with density $f_S(t)=\lambda e^{-t\lambda}$.

Hilbert Spaces

Projection Theorem

E,S Hilbert spaces with $S\subset E$, then

$$\forall v\in E\,,\exists!\ b\in S\ |$$

$$b=\arg\min_{c\in S}\|v-c\|\,,\ \langle v-b,c\rangle=0\,,\forall\ c\in S\,,$$

Projection Theorem w.s.s.

$E\ S$ Hilbert spaces of w.s.s. processes with $S\subset E\subseteq\mathrm{L}^2(P)$, then

$$\forall X[n]\in E\,,\exists!\ Y[n]\in S\ |$$

$$Y[n]=\arg\min_{U[n]\in S}\mathrm{E}[|X[n]-U[n]|^2]\,,$$

$$\mathrm{E}[(X[n]-Y[n])U^*[n]]=0\,,\forall\ U[n]\in S\,,$$

Empirical Statistics

Bias & Variance

$\widehat{S}(x[1],\ldots,x[N])$ empirical statistics of a probabilistic moment S.

Bias $\mathrm{E}\big[\widehat{S}(X[1],\ldots,X[N])\big]-S$,

Variance $\mathrm{Var}\big(\widehat{S}(X[1],\ldots,X[N])-S\big)$.

Unbiased & Biased Correlation

$$\widehat{R}_X^{\mathrm{NB}}(k)=\frac{1}{N-|k|}\sum_{n=1}^{N-|k|}x[n+k]x^*[n]\,,$$

$$\widehat{R}_X^{\mathrm{B}}(k)=\frac{1}{N}\sum_{n=1}^{N-|k|}x[n+k]x^*[n]\,.$$

Methods

Linear Estimation of w.s.s.: Wiener Filter

Estimation of $X[n]$ given $Y[n]$

Normal equations $R_{XY}[u]=\sum_{m\in\mathbb{Z}}h[m]R_Y[u-m]$

Wiener Filter $H(e^{j\omega})=\frac{S_{XY}(\omega)}{S_Y(\omega)}$

Linear Prediction of w.s.s.: Yule-Walker

Prediction of $X[n]$ as linear combination of $X[n-1],\ldots,X[n-N]$.

Coefficients a_k solutions of

$$\begin{bmatrix}R_X[0]&\ldots&R_X[N-1]\\\vdots&\ddots&\vdots\\R_X[N-1]&\ldots&R_X[0]\end{bmatrix}\begin{bmatrix}a_1\\\vdots\\a_N\end{bmatrix}=\begin{bmatrix}R_X[1]\\\vdots\\R_X[N]\end{bmatrix}.$$

Linear Estimation of AR: Yule-Walker

$\sum_{k=0}^{N}p_kX[n-k]=W[n]$. Coeff. p_k solution of

$$-\begin{bmatrix}R_X[0]&\ldots&R_X[N-1]\\\vdots&\ddots&\vdots\\R_X[N-1]&\ldots&R_X[0]\end{bmatrix}\begin{bmatrix}p_1\\\vdots\\p_N\end{bmatrix}=\begin{bmatrix}R_X[1]\\\vdots\\R_X[N]\end{bmatrix}.$$

$$\sigma_W^2=R_X[0]+R_X[1]p_1+\ldots+R_X[N]p_N$$

Linear Pretiction of AR: Projection Theorem

$H(X,n)=H(W,n)\,,\quad\forall\ n\in\mathbb{Z}\,.$

Intuitive property:

$$Y\in H(X,n+k)\,,\ Y=A+B\,,\ A\perp H(X,n)\,,\ B\in H(X,n)$$

orthogonal projection of Y onto $H(X,n)$ is B.

Estimation Param. Prob.: MLE

$\boldsymbol{\theta}$ parameters of the prob. function f_Y, $y[1],\ldots,y[N]$ realization of the process $Y[n]$, then

$$\widehat{\boldsymbol{\theta}}=\arg\max_{\boldsymbol{\theta}}f_Y(y[1],\ldots,y[N],\boldsymbol{\theta})$$

Spectral Estimation

Periodogram: General w.s.s. process

$$P_X^N(\omega)=\frac{1}{N}\left|\sum_{n=1}^{N}x[n]e^{-j\omega n}\right|^2=\frac{1}{N}|\widehat{x}_N(\omega)|^2\,,$$

Bias $\sum_{k=-N+1}^{N-1}\frac{N-|k|}{N}R_X[k]e^{-j\omega k}-S_X(\omega)$

Variance constant

Resolution $\Delta f>\frac{1}{N}$

Annihilating Filter: Line Spectra

Estimation of line spectrum frequencies and amplitudes of a Harmonic w.s.s. process in absence of noise

1) Given $2K$ observations, solve the system

$$\begin{bmatrix}x[K-1]&\ldots&x[0]\\\vdots&&\vdots\\x[2K-2]&\ldots&x[K-1]\end{bmatrix}\begin{bmatrix}h[1]\\\vdots\\h[K]\end{bmatrix}=-\begin{bmatrix}x[K]\\\vdots\\x[2K-1]\end{bmatrix}$$

2) Compute $H(z)$ and the zeros of $H(z)$

3) Compute the argument of the zeros of $H(z)$

4) Compute ω_k from the zeros' arguments

5) Compute the amplitudes $|\alpha_k|^2$ by solving

$$\begin{bmatrix}1&\ldots&1\\e^{j\omega_1}&\ldots&e^{j\omega_K}\\\vdots&&\vdots\\e^{j\omega_1(K-1)}&\ldots&e^{j\omega_K(K-1)}\end{bmatrix}\begin{bmatrix}\alpha_1e^{j\Theta_1}\\\alpha_2e^{j\Theta_2}\\\vdots\\\alpha_Ke^{j\Theta_K}\end{bmatrix}=\begin{bmatrix}x[0]\\x[1]\\\vdots\\x[K-1]\end{bmatrix}$$

MUSIC: Line Spectra

Estimation of line spectrum frequencies and amplitudes of a Harmonic w.s.s. process in the presence of noise

1) Given M observations with $M>>N>K$ center the process and compute the empirical correlation matrix

$$\widehat{\boldsymbol{R}}_{\boldsymbol{Y}}^{NN}=\frac{1}{M-N+1}\sum_{n=1}^{M-N+1}\boldsymbol{y}^{N1}[n]\boldsymbol{y}^{N1}[n]^H\,;$$

2) Compute the eigendecomposition $\widehat{\boldsymbol{G}}^{N(N-K)}$ of $\widehat{\boldsymbol{R}}_{\boldsymbol{Y}}^{NN}$ corresponding to λ_{K+1}^R to λ_N^R.

3.a) Determine the peaks of

$$\frac{1}{\boldsymbol{e}^{N1}(\omega)^H\widehat{\boldsymbol{G}}^{N(N-K)}\widehat{\boldsymbol{G}}^{N(N-K)\,H}\boldsymbol{e}^{N1}(\omega)}\,;$$

where

$$\boldsymbol{e}^{N1}(\omega)=\begin{bmatrix}1&e^{-j\omega}&\ldots&e^{-j(N-1)\omega}\end{bmatrix}^T\,,$$

3.b) Determine the minimum values of

$$\boldsymbol{e}^{N1}(\omega)^H\widehat{\boldsymbol{G}}^{N(N-K)}\widehat{\boldsymbol{G}}^{N(N-K)\,H}\boldsymbol{e}^{N1}(\omega)\,.$$

4) Compute the modulus of the amplitudes using

$$\boldsymbol{R}_{\boldsymbol{Y}}^{NN}=\boldsymbol{E}^{NK}\boldsymbol{A}^{KK}\boldsymbol{E}^{NK\,H}+\sigma_W^2\boldsymbol{I}^{NN}\,.$$

Yule-Walker: Smooth Spectra

1) Given N observations with $N>>M$ center the process and

compute the empirical correlation

$$\widehat{R}_X[k]=\frac{1}{N}\sum_{n=1}^{N-k} x_0[n+k]x_0[n]^*\,,\qquad k=0,\ldots,M$$

2) Solve the Yule Walker equations to obtain $\hat{p}_1,\ldots,\hat{p}_M$

3) Compute the estimate of the spectrum as

$$\widehat{S}_X(\omega)=\left.\frac{\widehat{\sigma}_W^2}{|\widehat{P}(z)|^2}\right|_{z=e^{j\omega}}.$$

Mixture Models

Sequence of samples $\boldsymbol{y}=[y[1],\ldots,y[N]]$, sequence of corresponding classes $\boldsymbol{c}=[c[1],\ldots,c[N]]$

$$F_{\boldsymbol{Y}}(\boldsymbol{y})=\sum_{\boldsymbol{c}\in\mathcal{C}} \mathrm{P}(\boldsymbol{Y}\leq\boldsymbol{y},\boldsymbol{C}=\boldsymbol{c})=\sum_{\boldsymbol{c}\in\mathcal{C}} \mathrm{P}(\boldsymbol{Y}\leq\boldsymbol{y}\mid\boldsymbol{C}=\boldsymbol{c})\mathrm{P}(\boldsymbol{C}=\boldsymbol{c})$$
$$=\sum_{\boldsymbol{c}\in\mathcal{C}}\prod_{n=1}^{N}\mathrm{P}(Y[n]\leq y[n]\mid C[n]=c[n])\mathrm{P}(\boldsymbol{C}=\boldsymbol{c})$$

i.i.d. Mixtures

$$\mathrm{P}(\boldsymbol{C}=\boldsymbol{c})=\mathrm{P}(C[1]=c[1])\ldots\mathrm{P}(C[N]=c[N])=\pi_{c[1]}\cdots\pi_{c[N]}\,,$$

Markovian Mixtures

$$\mathrm{P}(\boldsymbol{C}=\boldsymbol{c})=\pi_{c[1]}p_{c[1]c[2]}\cdots p_{c[N-1]c[N]}\,,$$

Discrete value Process + Noise

$\boldsymbol{Y}=\boldsymbol{X}+\boldsymbol{W}$ where $\boldsymbol{X}$ is a discrete value process and $\boldsymbol{W}$ a white Gaussian noise.

$$f_{\boldsymbol{Y}}(\boldsymbol{y})=\sum_{\boldsymbol{x}\in\mathcal{C}}\prod_{n=1}^{N}\mathcal{G}_{x[n],\sigma^2}(y[n])\mathrm{P}(\boldsymbol{X}=\boldsymbol{x})\,.$$

Denoising a Discrete Value Process

Estimate the parameters of the mixture model using the maximum likelihood approach; Estimate the original signal using the maximum *a posteriori* approach, *i.e.*, find $\boldsymbol{x}$ maximizing the *a posteriori* distribution

$$\mathrm{P}(\boldsymbol{X}=\boldsymbol{x}\mid\boldsymbol{y})=\frac{f_{\boldsymbol{Y}}(\boldsymbol{y}\mid\boldsymbol{X}=\boldsymbol{x})\mathrm{P}(\boldsymbol{X}=\boldsymbol{x})}{f_{\boldsymbol{Y}}(\boldsymbol{y})}$$

PCA

Principal Components Computaiton

M data vectors, each characterized of N variables (realization of a zero mean w.s.s. process) $\boldsymbol{c}_m=[c_m[1],\ldots,c_m[N]]^T\,,\quad m=1,\ldots,M.$

Empirical correlation matrix

$\widehat{\boldsymbol{R}}_{\boldsymbol{c}}=\frac{1}{M}\sum_{m=1}^{M}\boldsymbol{c}_m*\boldsymbol{c}_m^H=\frac{1}{M}\boldsymbol{C}*\boldsymbol{C}^H\,,(N\times N)$, where $\boldsymbol{C}=\begin{bmatrix}\boldsymbol{c}_1 & \ldots & \boldsymbol{c}_M\end{bmatrix}$, V solution of the equation $\widehat{\boldsymbol{R}}_{\boldsymbol{c}}\boldsymbol{V}=\boldsymbol{V}\boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$ and $\boldsymbol{V}^H\widehat{\boldsymbol{R}}_{\boldsymbol{c}}\boldsymbol{V}=\boldsymbol{\Lambda}$.

Principal components $\boldsymbol{Z}=\boldsymbol{V}^T\boldsymbol{C}\,,(N\times M)$, uncorrelated.

Invertible transformation $\boldsymbol{C}=\boldsymbol{V}\boldsymbol{Z}\,,$

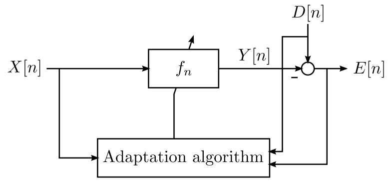
Analysis

$K<<N$ eigenvalues with highest values (lossy/lossless reduction of variables)

Adaptive Filtering / Echo cancellation

Wiener-Hopf equations $\sum_{k\in\mathbb{Z}}h[n-k]R_Y(k-l)=R_{XY}(n-l)\,,\quad\forall\, l\,.$

Echo cancellation setup



$$E[n]=D[n]-f_n*X[n]=S[n]+h*X[n]-f_n*X[n]=S[n]+(h-f_n)*X[n]\,.$$

Cost function & normal equations

Cost function for a k-tap filter

$$J(f_n)=\mathrm{E}\left[|E[n]|^2\right]=\mathrm{E}\left[|D[n]-f_n*X[n]|^2\right],\quad\text{w.r.t.}\quad f_n[l]\,,l=0,1,\ldots,K$$

Minimum of the cost function = normal equations

$$\sum_{l=0}^{K}f_n[l]R_X(n-l,n-i)=R_{DX}(n,n-i)\,,\quad \boldsymbol{R}_{X,n}\boldsymbol{f}_n=\boldsymbol{r}_{DX,n}\,.$$

Iterative solution

$$\boldsymbol{f}^{(i+1)}=\boldsymbol{f}^{(i)}+\mu\boldsymbol{p}\,,\quad i=0,1,\ldots,\quad \mu\ \&\ \boldsymbol{p}\ \text{such that}\ J(\boldsymbol{f}^{(i+1)})<J(\boldsymbol{f}^{(i)})$$

Convergence conditions

$$0<\mu<2/\lambda_{\max}\,,\quad \boldsymbol{p}=\tfrac{1}{2}(\boldsymbol{r}_{DX}-\boldsymbol{R}_X\boldsymbol{f}^{(i)})\ \text{or}\ \boldsymbol{p}=4\boldsymbol{R}_X^{-1}(\boldsymbol{r}_{DX}-\boldsymbol{R}_X\boldsymbol{f}^{(i)})(\text{Newton})$$

Convergence rate

- $0\leq 1-\mu\lambda_j<1$, monotonic decay to zero;
- $-1<1-\mu\lambda_j<0$ oscillatory decay to zero.
- $K+1$ modes $\{1-\mu\lambda_j, j=0,\ldots,K\}$. The modes with maximum magnitude (slowest rate of convergence), determine the convergence rate of the algorithm. One can select μ optimally by minimizing the value of the slowest mode $\min_\mu\max_{j=0,\ldots,K}|1-\mu\lambda_j|$, with the constraint that each of the modes is stable, i.e., $|1-\mu\lambda|<1$.

Computational burden reduction

Merging interation and adaptation

$$\boldsymbol{f}_{n+1}=\boldsymbol{f}_n+\mu(\boldsymbol{r}_{DX,n+1}-\boldsymbol{R}_{X,n+1}\boldsymbol{f}_n)\,,$$
$$\boldsymbol{f}_{n+1}=\boldsymbol{f}_n+\mu\boldsymbol{R}_{X,n+1}^{-1}(\boldsymbol{r}_{DX,n+1}-\boldsymbol{R}_{X,n+1}\boldsymbol{f}_n)\quad(\text{Newton})\,,$$

Replacing statistics with individual values

$$\boldsymbol{f}_{n+1}=\boldsymbol{f}_n+\mu\boldsymbol{X}_n(D[n]-\boldsymbol{X}_n^T\boldsymbol{f}_n)=\boldsymbol{f}_n+\mu\boldsymbol{X}_nE[n]\,,$$