Statistical Signal & Data Processing - COM500
Midterm Exam

April 7 2022, Duration 1h30

Read Me First!

You are allowed to use:

- A handwritten cheatsheet (1 A4 sheet, double sided) summarizing the most important formulas (no exercise text or exercise solutions);
- A pocket calculator.

You are definitively not allowed to use:

- Any kind of support not mentioned above;
- Your neighbor; Any kind of communication systems (smartphones etc.) or laptops;
- Printed material; Text and Solutions of exercises/problems; Lecture notes or slides.

Write solutions on separate sheets, *i.e.* no more than one solution per paper sheet.

Return your sheets ordered according to problem (solution) numbering.

All the best for your exam!!

Warmup Exercise

This is a warm up problem .. do not spend too much time on it. Please provide justified, rigorous, and simple answers. If needed, you can add assumptions to the problem setup.

Exercise 1. SPECTRAL POWER (5 POINTS)

Assume you want to compute the power (sum) of two specific frequency intervals $[50, 70]$ Hz and $[100, 120]$ Hz of a power spectral density $S_x[k]$ computed using the samples $n = 1, \ldots, 100000$ of a signal $x[n]$ (sampled at $f_s = 2\,\text{kHz}$).

We assume that the spectral density $S_x[k]$ is also composed of 100000 samples, that is, $k = 1, \ldots, 100000$.

The power P is then computed as a sum of the values of $S_x[k]$ corresponding to the two frequency intervals $[50, 70]$ Hz and $[100, 120]$ Hz, namely

$$P_{[50,70]} = \sum_{k=k_1}^{k_2} S_x[k]\,,$$
$$P_{[100,120]} = \sum_{k=k_3}^{k_4} S_x[k]\,,$$
$$P_{\text{tot}} = P_{[50,70]} + P_{[100,120]}\,.$$

1) Compute k_1, k_2, k_3, and k_4;
2) How can you exploit the fact the $S_x[k]$ is a two sided spectrum?

Solution 1.

1) Given a sampling frequency f_s and a number of samples N, the relation between frequencies $f \in [0, f_s)]$, normalized frequencies $\tilde{f} \in [0, 1)$, and indexes $k = 1, \ldots, N$ is

$$\tilde{f} = \frac{f}{f_s}\,, \qquad k = \tilde{f}N + 1 = \frac{f}{f_s}N + 1\,.$$

Therefore the intervals $[50, 70]$ Hz and $[100, 120]$ Hz, can be expressed in normalized frequencies as $[0.05, 0.07]$ and $[0.1, 0.12]$ and in terms of sample indexes $[5001, 7001]$ and $[10001, 12001]$.

2) We can make an average of the two symmetric part of the spectrum to reduce the estimation errors.

Main Problem

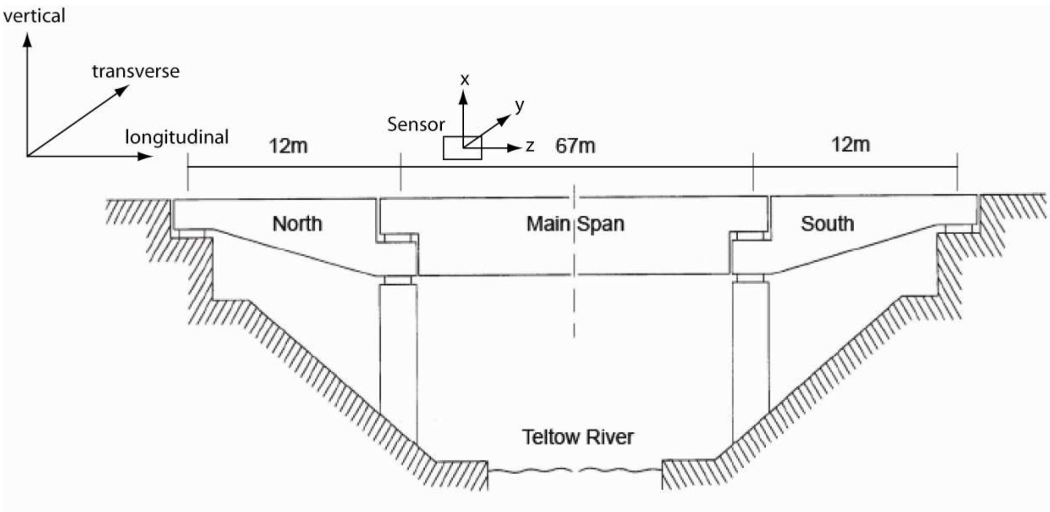
Here comes the core part of the exam .. take time to read the introduction and each problem statement. Please provide justified, rigorous, and simple answers. Remember that you are not simply asked to describe statistical signal processing tools, but you are rather asked to describe how to apply such tools to the given problem. If needed, you can add assumptions to the problem setup.

Exercise 2. Bridge Vibrations (30 points)

Bridge inspection for preventive maintenance is commonly carried on using traditional visual inspection tools. The latter can only detect obvious damages like disruption, cracks or rust on the surface of bridges.

An advanced non-destructive inspection method is based on vibration measurements obtained by mean on an accelerometer. Vibration monitoring can indeed immediately detect changes of structural integrity and even determine type and location of an occurred malfunction.

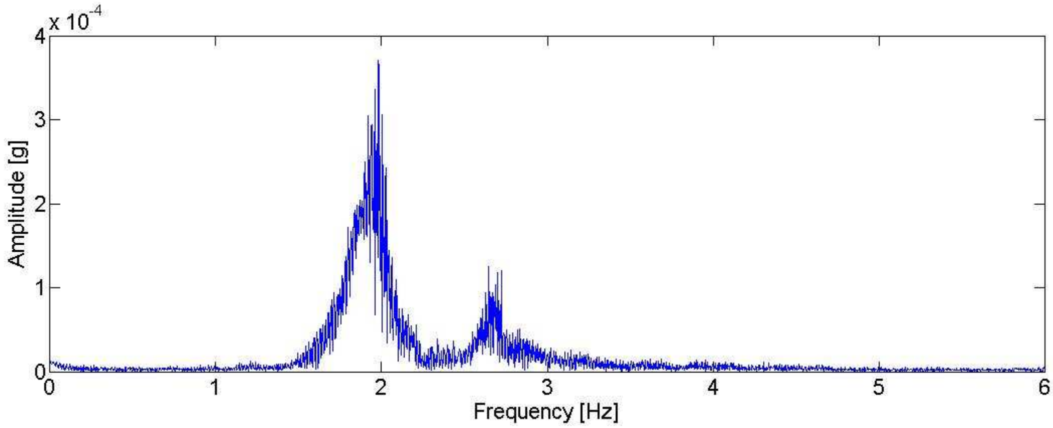
The picture below shows and application of 3D vibration measurement on the Komtur Bridge in Berlin, Germany.



Schematics of vibration measurement of the Komtur bridge in Berlin, Germany.

We denote with $x[n]$ the (real) signal of the vertical acceleration (corresponding to the x axis in the above picture). We suppose that we have obtained $N = 100000$ samples using a sampling frequency $f_s = 1000$ Hz.

The power spectral density $S_x[k]$ of the vibration measurement $x[n]$ is depicted in the figure below, we we can identify the very distinctive vibration characteristics of the Komtur bridge.



Power spectral $S_x[k]$ density of the vertical vibration measurement $x[n]$ of the Komtur bridge.

Please notice that:

- Being the signal real, only half of the spectrum is displayed;
- Only the relevant frequency interval is presented;
- The constant (DC) component, due to the gravity, has been eliminated, *i.e.*, the signal $x[n]$ has been centred.

The characteristic "natural" vibration frequencies of the bridge are around $f_a = 2\,\text{Hz}$ and $f_b = 2.6\,\text{Hz}$.
We now would like to develop an automatic vibration detection system that shall work for any general bridge.

Please notice that questions 1), 2) & 3), 4), 5) are independent.

Consider the spectrum $S_x[k]$ of the vibration measurement $x[n]$ of the Komtur bridge presented above. Suppose we have obtained the above spectrum with a periodogram computed using the 100000 samples of $x[n]$ (sampled at $f_s = 1\,\text{kHz}$).

1) Is it possible that the bridge under test has more than 2 main vibration frequencies? If so, what are the possible values of such frequencies?

We now proceed to the testing of another bridge, and we suppose, for simplicity, that it has 3 "natural" vibration frequencies f_1, f_2, and f_3 (in hertz).

Due to interferences, the vibration measurement, that we shall now denote with $y[n]$, is affected by a white Gaussian noise.

2) Propose a w.s.s. model for the vibration measurement $y[n]$.

3) Prove that the model you propose is indeed w.s.s..

We now would like to estimate the three frequencies f_1, f_2, and f_3 (in hertz).

4) Propose a parametric method to estimate the 3 main frequencies (only the frequencies!) of the vibration measurement $y[n]$. Precisely describe such method: You are given the 100000 samples, sampled at $f_s = 1\,\text{kHz}$, and you are asked to detail each step as if you have to implement the method in a computer. Precisely indicate the input and output of each step.

Recent studies showed that bridges vibrate not at specific frequencies but rather at specific frequency intervals (around main frequencies). We can indeed see such a behaviour by noticing that the experimental power spectral density depicted in the picture above shows pulses rather than lines.

To compute such pulses, it is necessary to estimate the whole power spectral density $S_y[k]$ and not only specific frequencies.

5) Propose another parametric method to estimate the shape of power spectral density (pulses) $S_y[k]$ of the the vibration measurement $y[n]$. Precisely describe such method: You are given the 100000 samples, sampled at $f_s = 1\,\text{kHz}$, and you are asked to detail each step as if you have to implement the method in a computer. Precisely indicate the input and output of each step.

Solution 2.

$N = 100000$, $f_s = 1\,\text{kHz}$, $f_a = 2\,\text{Hz}$ and $f_b = 2.6\,\text{Hz}$.

1) The spectral resolution of the periodogram is, $\Delta f = f_s/N = 1\,\text{kHz}/100000 = 0.01\,\text{Hz}$. Any other spectral line falling in the interval $[f_k - \Delta f, f_k + \Delta f]$, $k = a, b$ will be "hidden" by the impulse shape of the main frequency f_k, $k = a, b$.

   Therefore, it is possible to have other vibration frequencies belonging to the intervals $[2 - 0.01, 2 + 0.01]\,\text{Hz}$, and $[2.6 - 0.01, 2.6 + 0.01]\,\text{Hz}$.

2) We have a real harmonic signal of 3 frequencies plus noise, that can be modelled as

$$Y[n] = X[n] + W[n] = \sum_{k=1}^{3} \alpha_k (e^{i(2\pi \frac{f_k}{f_s} n + \Theta_k)} + e^{-i(2\pi \frac{f_k}{f_s} n + \Theta_k)}) + W[n]\,,$$
$$= \sum_{k=1}^{3} \alpha_k (e^{i2\pi \frac{f_k}{f_s} n} e^{i\Theta_k} + e^{-i2\pi \frac{f_k}{f_s} n} e^{-i\Theta_k}) + W[n]\,,$$

   where Θ_k are i.i.d. uniform (on $[0, 2\pi]$) random variables, and $W[n]$ is a centered white noise.

3) We need to prove that:
   $\mathrm{E}[Y[n]] = \text{constant}$

$$\mathrm{E}[Y[n]] = \mathrm{E}\left[\sum_{k=1}^{3} \alpha_k (e^{i2\pi \frac{f_k}{f_s} n} e^{i\Theta_k} + e^{-i2\pi \frac{f_k}{f_s} n} e^{-i\Theta_k}) + W[n]\right]$$
$$= \sum_{k=1}^{3} \alpha_k (e^{i2\pi \frac{f_k}{f_s} n} \mathrm{E}\left[e^{i\Theta_k}\right] + e^{-i2\pi \frac{f_k}{f_s} n} \mathrm{E}\left[e^{-i\Theta_k}\right]) + \mathrm{E}[W[n]]\,.$$

Notice that $\mathrm{E}\left[e^{\pm i\Theta}\right] = \int_0^{2\pi} e^{\pm i\theta}\frac{1}{2\pi}d\theta = 0$ and $\mathrm{E}\left[W[n]\right] = 0$. Therefore $\mathrm{E}\left[Y[n]\right] = 0$

$$\mathrm{E}\left[Y[n+l]Y[n]^*\right] = R_Y[l].$$

$$\begin{aligned}
\mathrm{E}\left[Y[n+l]Y[n]^*\right] =& \mathrm{E}\left[\left(X[n+l]+W[n+l]\right)\left(X[n]+W[n]\right)^*\right] \\
=& \mathrm{E}\left[X[n+l]X[n]^*\right] + \mathrm{E}\left[W[n+l]X[n]^*\right] + \mathrm{E}\left[X[n+l]W[n]^*\right] + \mathrm{E}\left[W[n+l]W[n]^*\right] \\
\overset{X\perp W}{=}& \mathrm{E}\left[X[n+l]X[n]^*\right] + \underbrace{\mathrm{E}\left[W[n+l]\right]}_{=0}\mathrm{E}\left[X[n]^*\right] + \mathrm{E}\left[X[n+l]\right]\underbrace{\mathrm{E}\left[W[n]^*\right]}_{=0} + \underbrace{\mathrm{E}\left[W[n+l]W[n]^*\right]}_{\sigma_W^2\delta_l} \\
=& \mathrm{E}\left[X[n+l]X[n]^*\right] + \sigma_W^2\delta_l\,.
\end{aligned}$$

Let's now take a look at the term $\mathrm{E}\left[X[n+l]X[n]^*\right]$. For notation ease, denote $\alpha_k e^{i2\pi\frac{f_k}{f_s}n} = \alpha_k e_k^+[n]$ and $\alpha_k e^{-i2\pi\frac{f_k}{f_s}n} = \alpha_k e_k^-[n]$

$$\begin{aligned}
\mathrm{E}\left[X[n+l]X[n]^*\right] =& \mathrm{E}\left[\sum_{k=1}^{3}\sum_{m=1}^{3}(e_k^+[n+l]e^{i\Theta_k} + e_k^-[n+l]e^{-i\Theta_k})(e_m^-[n]e^{-i\Theta_m} + e_m^+[n]e^{i\Theta_m})\right] \\
=& \sum_{k=1}^{3}\sum_{m=1}^{3}\mathrm{E}\left[(e_k^+[n+l]e^{i\Theta_k} + e_k^-[n+l]e^{-i\Theta_k})(e_m^-[n]e^{-i\Theta_m} + e_m^+[n]e^{i\Theta_m})\right]
\end{aligned}$$

Notice that

$$\begin{aligned}
&\sum_{k=1}^{3}\sum_{m=1,m\neq k}^{3}\mathrm{E}\left[(e_k^+[n+l]e^{i\Theta_k} + e_k^-[n+l]e^{-i\Theta_k})(e_m^+[n]e^{i\Theta_m} + e_m^-[n]e^{-i\Theta_m})\right] \\
&= \sum_{k=1}^{3}\sum_{m=1,m\neq k}^{3}\mathrm{E}\left[(e_k^+[n+l]e^{i\Theta_k} + e_k^-[n+l]e^{-i\Theta_k})\right]\mathrm{E}\left[(e_m^+[n]e^{i\Theta_m} + e_m^-[n]e^{-i\Theta_m})\right] = 0\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathrm{E}\left[X[n+l]X[n]^*\right] \\
&= \sum_{k=1} x\mathrm{E}\left[(e_k^+[n+l]e^{i\Theta_k} + e_k^-[n+l]e^{-i\Theta_k})(e_k^+[n]e^{i\Theta_k} + e_k^-[n]e^{-i\Theta_k})\right] \\
&= \sum_{k=1}^{3}\left(e_k^+[n+l]e_k^+[n]\mathrm{E}\left[e^{i2\Theta_k}\right] + e_k^-[n+l]e_k^+[n]\mathrm{E}\left[e^0\right] + e_k^+[n+l]e_m^-[n]\mathrm{E}\left[e^0\right] + e_k^-[n+l]e_m^-[n]\mathrm{E}\left[e^{-i2\Theta_k}\right]\right) \\
&= \sum_{k=1}^{3}\left(e_k^-[n+l]e_k^+[n] + e_k^+[n+l]e_k^-[n]\right) = \sum_{k=1}^{3}\left(\alpha_k^2 e^{-i2\pi\frac{f_k}{f_s}(n+l)}e^{i2\pi\frac{f_k}{f_s}n} + \alpha_k^2 e^{i2\pi\frac{f_k}{f_s}(n+l)}e^{-i2\pi\frac{f_k}{f_s}n}\right) \\
&= \sum_{k=1}^{3}\alpha_k^2(e^{-i2\pi\frac{f_k}{f_s}l} + e^{i2\pi\frac{f_k}{f_s}l})\,.
\end{aligned}$$

Finally

$$\mathrm{E}\left[Y[n+l]Y[n]^*\right] = \sum_{k=1}^{3}\alpha_k^2(e^{-i2\pi\frac{f_k}{f_s}l} + e^{i2\pi\frac{f_k}{f_s}l}) + \sigma_W^2\delta_l\,,$$

obtaining an expression depending only on l.

4) Given the presence of noise and that fact that we aim at the estimation of the vibration frequencies, *i.e.* the position of spectral lines, MUSIC is a very good candidate method.

Notice that 3 frequencies of a real harmonic signal correspond to 6 spectral lines (3 positive and 3 symmetrically negative frequencies).

$N = 100000$, samples $y[1], \ldots, y[100000]$, $f_s = 1\,\text{kHz}$.

- Center the process $\tilde{y}[n] = y[n] - m_Y$, where $m_Y = \frac{1}{100000}\sum_{k=1}^{100000} y[k]$;
- Compute the biased empirical correlation

$$\widehat{R}_Y[k] = \frac{1}{100000}\sum_{n=1}^{100000-k} \tilde{y}[n+k]\tilde{y}[n]\,, \quad k = 0, \ldots, (100000-1)\,, \quad \widehat{R}_Y[-k] = \widehat{R}_Y[k]\,.$$

  Set $8 << M << 100000$, The empirical correlation matrix is then given by

$$\widehat{\boldsymbol{R}}_Y^{M\times M} = \begin{bmatrix} \widehat{R}_Y[0] & \widehat{R}_Y[1] & \cdots & \widehat{R}_Y[M-1] \\ \widehat{R}_Y[-1] & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \widehat{R}_Y[-M+1] & \cdots & \cdots & \widehat{R}_Y[0] \end{bmatrix}.$$

  Notice that we set M bigger than the number $K = 6$ of the positons of the spectral lines we are looking for (so to exploit redundancy for the estimation of the positions), and smaller than the number of samples (so to reduce the extreme lag issues of the correlation)
- Compute the M eigenvalues $\boldsymbol{\lambda}$ and M eigenvectors $\boldsymbol{g}$ of $\widehat{\boldsymbol{R}}_X^{M\times M}$.
- Call $\boldsymbol{G}^{M\times(M-6)}$ the matrix of the $M-6$ eigenvectors corresponding to the $M-6$ smaller eigenvalues.
- Define the vector $\boldsymbol{e}^{M\times 1}(\omega) = \begin{bmatrix}1 & e^{-j\omega} & \ldots & e^{-j(M-1)\omega}\end{bmatrix}^T$ as a function of the variable ω.
- Find the 6 values of ω $(\omega_1, \ldots, \omega_6)$ minimizing the equation

$$\boldsymbol{e}^{M\times 1}(\omega)^H \widehat{\boldsymbol{G}}^{M\times(M-6)} \widehat{\boldsymbol{G}}^{M\times(M-6)^H} \boldsymbol{e}^{M\times 1}(\omega)\,.$$

- Obtain the 6 normalized frequencies as $\tilde{f}_k = \frac{\omega_k}{2\pi}$. Out of the so obtained 6 frequencies, select the 3 that fall in $[0, 0.5)$.

5) Here we need to estimate the shape of the spectrum which is realistically composed of "pulses". Consequently the optimal approach is to estimate the spectrum as a smooth spectrum

$$S_Y(\omega) = \frac{\sigma^2}{|P(e^{j\omega})|^2} = \frac{\sigma^2}{|1 + p_1 e^{-j\omega} + \ldots + p_M e^{-j\omega M}|^2}$$

assuming the signal to be an AR process. Here, the AR process should be at least of order $M = 6$ in order to account for the 6 peaks of the spectrum shape (3 vibration mode of a real signal).

The polynomial coefficients $p_1, \ldots, p_6$ and σ^2 are estimated using the Yule-Walker equations.

- Center the process $\tilde{y}[n] = y[n] - m_Y$, where $m_Y = \frac{1}{100000}\sum_{k=1}^{100000} y[k]$;
- Compute the empirical biased correlation

$$\widehat{R}_Y[k] = \frac{1}{100000}\sum_{n=1}^{100000-k} \tilde{y}[n+k]\tilde{y}[n]\,, \quad k = 0, \ldots, (100000-1)\,, \quad \widehat{R}_Y[-k] = \widehat{R}_Y[k]\,.$$

- Write the Yule Walker equations for the estimation of $p_1, \ldots, p_6$ and σ^2

$$\begin{bmatrix} \widehat{R}_Y(0) & \ldots & \widehat{R}_Y(5) \\ \vdots & \ddots & \vdots \\ \widehat{R}_Y(5) & \ldots & \widehat{R}_Y(0) \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_6 \end{bmatrix} = - \begin{bmatrix} \widehat{R}_Y(1) \\ \vdots \\ \widehat{R}_Y(6) \end{bmatrix}$$

$$\sigma^2 = \widehat{R}_Y(0) + \widehat{R}_Y(1)p_1 + \ldots + \widehat{R}_Y(6)p_6\,.$$

- Solve the equations (Toeplitz symmetric 6^2+6 multiplications, or use the Levinson's algorithm) in order to obtain $p_1, \ldots, p_6$ and σ^2
- Compute the estimation of the spectrum as

$$\widehat{S}_Y(\omega) = \frac{\sigma^2}{|1 + p_1 e^{-j\omega} + \ldots + p_6 e^{-j\omega 6}|^2}$$

  where the frequency (in radiants) ω can be discretized over $N = 100000$ points: $\omega = 2\pi k/N$, $k = 0, \ldots, N-1$

$$\widehat{S}_Y(k) = \frac{\sigma^2}{\left|1 + p_1 e^{-j2\pi\frac{k}{N}} + \ldots + p_6 e^{-j2\pi\frac{k}{N}6}\right|^2}\,, \quad k = 0, \ldots, N-1\,.$$