Statistical Signal Processing

Midterm Exam

Thursday, 21 April 2016

You will hand in this sheet together with your solutions.

Write your personal data (please make it readable!).

Seat Number:

Family Name:

Name:

Read Me First!

Only the personal cheat sheet is allowed.
No class notes, no exercise text or exercise solutions.

Write solutions on separate sheets,
***i.e.* no more than one solution per paper sheet.**

Return your sheets ordered according to problem (solution) numbering.

Return the text of the exam.

Warmup exercises

This is a warm up problem .. do not spend too much time on it.

Please provide justified, rigorous, and simple answers.

Exercise 1. PROBABILITY (3 PTS)

Let X a continuos valued random variable uniformly distributed over [0,10], and Y a discrete valued random variable, taking integer values between 0 and 10 (both included), each with equal probability, *i.e.*, $\mathrm{P}(Y = l) = \mathrm{P}(Y = k)$, for all l,k=0,1,2,3, . . . ,9,10.

1) Give the expression of the probability density function $f_X(x)$ of X;
2) Give the expression $\mathrm{P}(3 \leq Y \leq 4)$;
3) Give the value of $\mathrm{P}(X = 5)$.

Exercise 2. PROBABILITIES AND STOCHASTIC PROCESSES (3PTS)

These are simple true/false questions, each counting 0.6 points.
NOTE: Don't answer randomly: Each wrong answer will count for -0.6 points!

Let X, Y, Z be continuous-valued random variables. Without further conditions, the following statements are true or false?

1) If X admits a probability density function $f_X(a)$, then $\mathbb{P}(X = a) = f_X(a)$.
2) If $Z = X + Y$, the cumulative distribution function of Z can be derived by convolution, i.e., $F_Z(a) = F_X(a) * F_Y(a)$.
3) If $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$, then X and Y are independent.

Let $X[n]$, $Y[n]$ and $Z[n]$ be a stochastic processes. Without further conditions, the following statements are true or false?

4) If $\mathbb{E}[X[n]] = 1$ and $\mathbb{E}[X[k]X[l]^*] = k - 2l$, then $X[n]$ is wide sense stationary.
5) If $X[n]$ and $Y[n]$ are i.i.d. Poisson distributed with mean λ, $Z[n] = X[n] + Y[n]$ has Poisson distribution with mean 2λ.

Exercise 3. HOW MANY PARAMETERS? (2 PTS)

Let $X[n]$, be a stochastic process taking 10 possible values, and such that

- $\mathrm{P}(X[n] = i_0 | X[n-1] = i_1, \ldots, X[n-k] = i_k) = \mathrm{P}(X[n] = i_0 | X[n-1] = i_1)$, for every n, k
- $\mathrm{P}(X[n] = i) = \mathrm{P}(X[n+l] = i)$ for every l

We would like to describe the process $X[n]$ for $n = 1, \ldots, 2000$, that is, we would like to characterize the law (probability) of $X[1], \ldots, X[2000]$. How many parameters are necessary in order to characterize $X[1], \ldots, X[2000]$?

Main exercises

Here comes the core part of the exam .. take time to read the introduction and each problem statement.

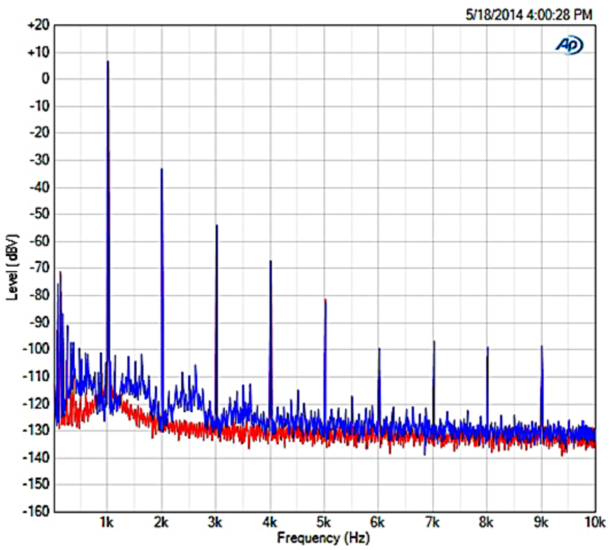
Please provide justified, rigorous, and simple answers.

Exercise 4. GUITARISTS LOVE TUBE AMPLIFIERS! (22PTS)
Tube amplifiers are definitively the preferred amplifiers among guitarists. They have a very particular sound, and an undoubtedly fascinating look.

Their particular sound is mostly due to the distortion they introduce. That is, if the input signal is a sinusoid, the output will not be a perfect sinusoid but a distorted one.

In the spectral domain, the distortion creates harmonics: If the input sinusoids has frequency f_0, then the spectrum of the output signal will be composed of the fundamental frequency (a Dirac at f_0) plus the harmonics (Diracs at frequencies $f_n = (n+1) \times f_0$, with $n = 1, 2, \ldots$). This suggests that the output signal can be seen as a sum of sinusoids.

To give an idea, the picture below depicts an example of part of the spectrum of an output signal (left and right channel) when the input is a sinusoidal signal with frequency 1KHz.



We now proceed in modeling an analyzing the output signal of a tube amplifier. We assume that:

- The input signal is a real sinusoid with frequency $4KHz$;

- The output signal is real;
- The output signal is sampled at $f_s = 40KHz$;
- Previous to sampling, the output signal is filtered with an anti-aliasing filter exactly dimensioned for the given sampling frequency;
- The output signal is noise-free.

We would like to model the sampled output signal as a w.s.s. stochastic process $Y[n]$.

1) Provide a mathematical expression for $Y[n]$ (sum of real sinusoids), describing it as w.s.s..
2) Check (prove) that such expression corresponds indeed to a w.s.s. stochastic process.

In order to verify our assumption we measure the output, therefore obtaining realizations $y[n]$ of a w.s.s. process, and we proceed to its spectral analysis.
As commonly done, as a first approach to spectral analysis we compute the periodogramm of the measured samples $y[n]$.

3) How many samples of the output signal $y[n]$ do we need in order to be able to distinguish its fundamental frequency and harmonics?

We know that a parametric spectral estimation approach can perform better than the periodogramm.

4) Propose an optimal parametric method in order to estimate the fundamental frequency and the harmonics of the output signal. Justify your choice and state your assumptions.
5) Precisely describe such method (you are asked to detail each step as if you have to implement the method in a computer).
6) How many samples $y[n]$ do we need in order to estimate the spectrum.

We now change the input to the sum of two real sinusoids, one with frequency $4KHz$, and the other at $7.5KHz$. The output will now be composed of two fundamental frequencies and the corresponding harmonics.
We start with the periodogramm.

7) How many samples of the output signal do we need in order to be able to distinguish the two fundamental frequencies and the corresponding harmonics?

Once again, we move on to a parametric method.

8) How should the optimal parametric method be modified in order to cope with the new input?
9) How many samples $y[n]$ do we need in order to estimate the spectrum.

And to conclude:

10) Compare the pros and cons of the periodogramm and the optimal parametric estimation method for the particular problem you had to solve in this exercise.