Statistical Signal Processing

Midterm Exam

Thursday, 16 April 2015

You will hand in this sheet together with your solutions.

Write your personal data (please make it readable!).

Seat Number:

Family Name:

Name:

Read Me First!

Only the personal cheat sheet is allowed.
No class notes, no exercise text or exercise solutions.

Write solutions on separate sheets,
***i.e.* no more than one solution per paper sheet**.

Return your sheets ordered according to problem (solution) numbering.

Return the text of the exam.

Warmup exercises

This is a warm up problem .. do not spend too much time on it.

Please provide justified, rigorous, and simple answers.

Exercise 1. AVERAGING PERIODOGRAM (3PTS)
The signal $X[n]$ is a zero mean Gaussian white noise with variance σ^2. We have measured N points of $X[n]$, and we use the periodogram $P_X^N(\omega)$ to estimate the power spectral density. Now we split the measured signal $(x[1], x[2], \cdots, x[N])$ into two parts

$$y_1 = (x[1], \cdots, x[N/2]), \quad y_2 = (x[N/2+1], \cdots, x[N]).$$

We denote the periodograms of these two parts as $P_{Y_1}(\omega)$ and $P_{Y_2}(\omega)$, respectively. Then we compute the average of these two periodograms

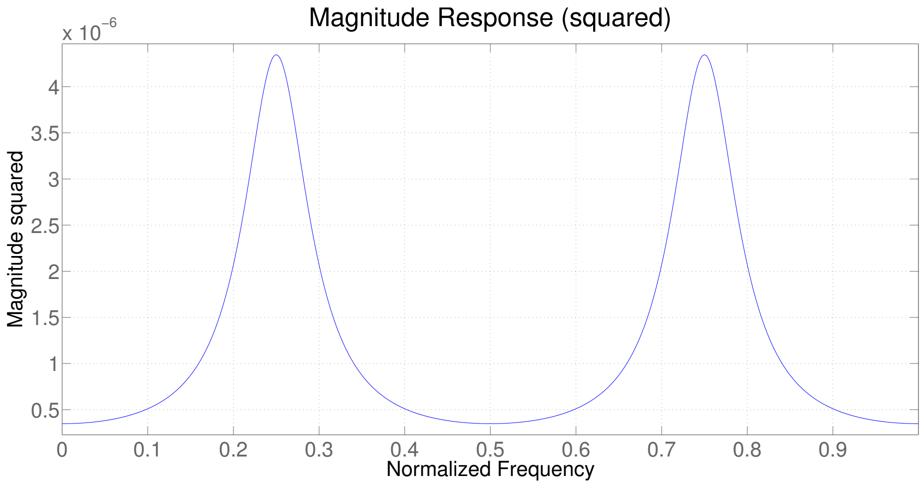
$$Q(\omega) = \frac{1}{2}(P_{Y_1}(\omega) + P_{Y_2}(\omega)).$$

$Q(\omega)$ provides a new estimator of the PSD of $X[n]$. What is the variance of this estimator $\mathrm{Var}(Q(\omega))$? Precisely justify your answer.

Solution 1.
Given that Y_1 and Y_2 are independent, $P_{Y_1}(\omega)$ computed substituting y_1 with Y_1, and $P_{Y_2}(\omega)$ computed substituting y_2 with Y_2, will also be independent. So the variance of the sum is the sum of the variances. By recalling that the variance of a periodogram of a white noise is given by the variance of the white noise, we have

$$\begin{aligned}\mathrm{Var}\left(Q(\omega)\right) &= \frac{1}{4}\left(\mathrm{Var}\left(P_{Y_1}(\omega)\right) + \mathrm{Var}\left(P_{Y_2}(\omega)\right)\right) \\ &= \frac{1}{4}\left(\sigma^2 + \sigma^2\right) \\ &= \frac{1}{2}\sigma^2 .\end{aligned}$$

Exercise 2. A FILTER (4PTS)
Consider a linear time-invariant filter. The plot below depicts the square magnitude of its transmittance $H(e^{j\omega})$ that has been computed as the square module of the discrete time Fourier transform of the impulse response $h[n]$.



Based on the given square magnitude is the filter

1) stable or instable?

2) causal or anti-causal?

Please justify precisely each answer.

3) Draw poles and zeros accordingly.

Solution 2.

1) The transmittance has been computed as the DTFT of the impulse response $h[n]$. In order to be defined (to exist), the DTFT requires that $h[n] \in \ell^1$. Therefore the filter is stable.

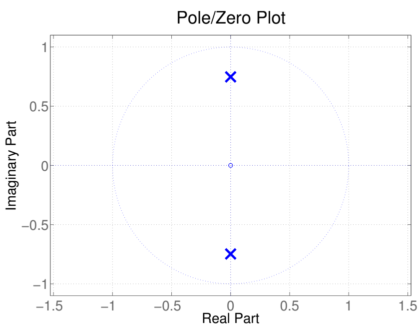
2) The filter can be

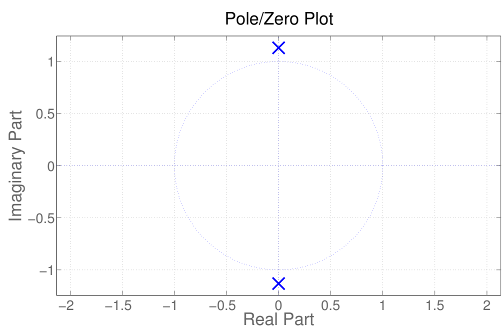
   – Causal, with poles inside the unit cercle

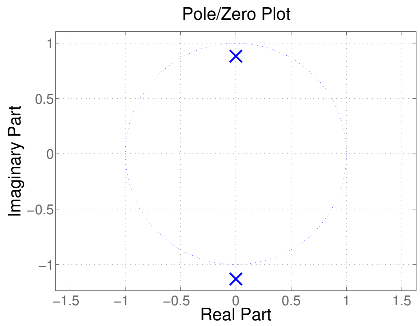
   – Anti-causal, with poles outside the unit circle

   – Non-causal (sum of a causal part and an anti-causal part), with poles inside the unit cercle for the causal part and poles outside for the anti-causal part.

3) If we consider, respectively, the causal, the anti-causal, and the non causal cases, the poles are as follows






Pole/Zero Plot
Imaginary Part
Real Part
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1
1.5

Main exercises

Here comes the core part of the exam .. take time to read the introduction and each problem statement.

Please provide justified, rigorous, and simple answers.

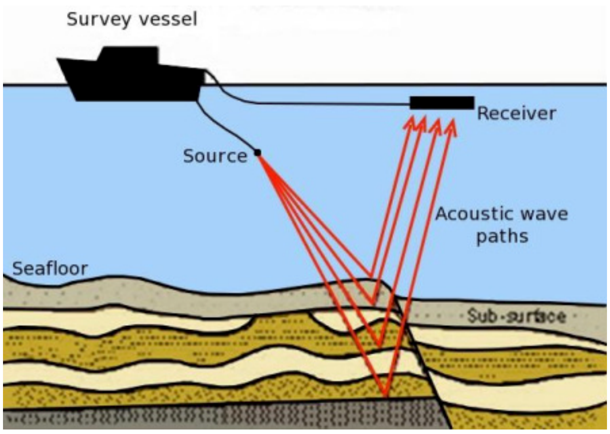
Exercise 3. Geophysical exploration (20pts)

Acoustic signal are used in geophysical exploration in order to study the geological interfaces in the subsurface of the earth.

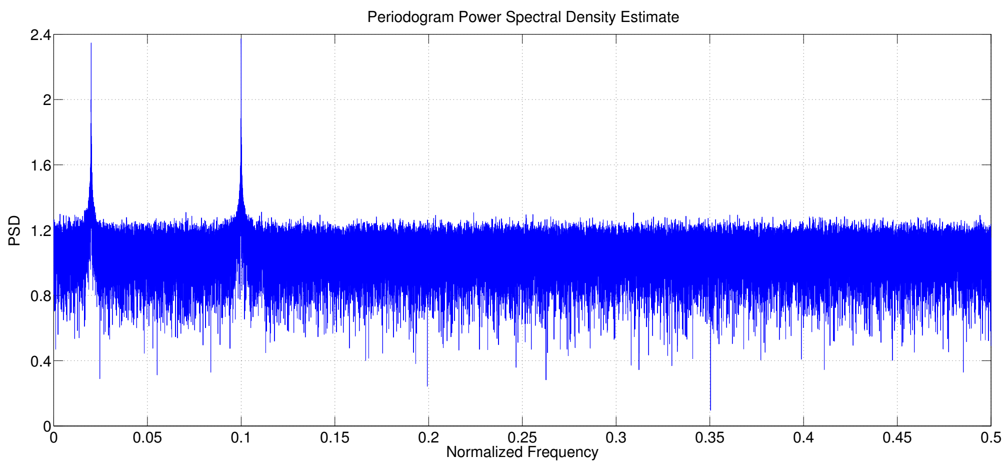
More precisely, sinusoidal waves are created by a controlled source and propagated through the subsurface of the earth. These will return to the surface after reflection at geological interfaces. Microphones distributed along the surface detect the returning waves which are then recorded in order to be processed and analyzed.

Notice that the sinusoidal wave signal is **real**.

The figure below depict an example of geophysical exploration of the seafloor.



We star by analyzing the data of a geophysical exploration session where one of the microphones of the detector has recorded 100000 **real** samples $X[1], \ldots, X[100000]$ at 20KHz. The corresponding periodogram S_X, between 0 and 0.5 (in normalized frequencies) reads



Notice that such a spectrum, between 0 and 0.5 (in normalized frequencies), presents two spikes at 0.02 and 0.1 (normalized frequencies) and a constant baseline.

1) What kind of signal has generated such spectrum? (it can be the sum of different signals, and do not forget that we have real signals here!)

The spectrum suggests that the source has generated **at least** 2 tones

2) Recalling that the sampling frequency is 20KHz, and that the two spikes are, in normalized frequencies, at 0.02 and 0.1, provide the frequencies in Hz of the two tones.

3) Recalling that we have recorded 100000 samples, is it possible that the source generated more than two tones? If yes, what are the possible frequencies of the additional tones? if no, why?

We now process the data of another geophysical exploration session where the recording has lasted 1 second (here again the data values are real). The sampling frequency is the same (20KHz), but we do not know exactly how many tones the source has generated. Nevertheless we assume that at maximum it has generated 5 different tones and that the recording has no noise (now we have used a very good microphone).

4) Propose a parametric method to estimate the frequencies (only the frequencies!) of the generated tones and justify you choice.

5) Precisely describe such method: You are given the samples corresponding to a recording of 1 second at 20KHz and you are asked to detail each step as if you have to implement the method in a computer.

6) What happens if the recording is corrupted by noise? Which of the steps that you have precisely described is affected?