Statistical Signal Processing

Midterm Exam

Thursday, 3 April 2014

You will hand in this sheet together with your solutions.

Write your personal data (please make it readable!).

Family Name:

Name:

E-mail:

Read Me First!

Only the personal cheat sheet is allowed.
No class notes, no exercise text or exercise solutions.

Write solutions on separate sheets,
***i.e.* no more than one solution per paper sheet**.

Return your sheets ordered according to problem (solution) numbering.

Return the text of the exam.

Warmup exercises

This is a warm up problem .. do not spend too much time on it.

Please provide justified, rigorous, and simple answers.

Exercise 1. Averaging Periodogram (3pts)
The signal $X[n]$ is a zero mean Gaussian white noise with variance σ^2. We have measured N points of $X[n]$, and would like to use the periodogram $P_X^N(\omega)$ to estimate the power spectrum density (PSD).
Now we split the measured signal $(x[1], x[2], \cdots, x[N])$ into two parts

$$y_1 = (x[1], \cdots, x[N/2]), \quad y_2 = (x[N/2+1], \cdots, x[N]).$$

We denote the periodograms of these two parts as $P_{Y_1}(\omega)$ and $P_{Y_2}(\omega)$, respectively. Then we compute the average of these two periodograms

$$Q(\omega) = \frac{1}{2}(P_{Y_1}(\omega) + P_{Y_2}(\omega)).$$

$Q(\omega)$ provides a new estimator of the PSD of $X[n]$. What is the variance of this estimator $\mathrm{Var}(Q(\omega))$? Precisely justify your answer by rigorously compute the variance of such an empirical estimator.
(*Hint: Y_1 and Y_2 are independent with each other.*)

Solution 1.
Given that Y_1 and Y_2 are independent, $P_{Y_1}(\omega)$ computed substituting y_1 with Y_1, and $P_{Y_2}(\omega)$ computed substituting y_2 with Y_2, will also be independent. So the variance of the sum is the sum of the variances. By recalling that the variance of a periodogram of a white noise is given by the variance of the white noise, we have

$$\begin{aligned}\mathrm{Var}\left(Q(\omega)\right) &= \frac{1}{4}\left(\mathrm{Var}\left(P_{Y_1}(\omega)\right) + \mathrm{Var}\left(P_{Y_2}(\omega)\right)\right) \\ &= \frac{1}{4}\left(\sigma^2 + \sigma^2\right) \\ &= \frac{1}{2}\sigma^2 .\end{aligned}$$

Main exercises

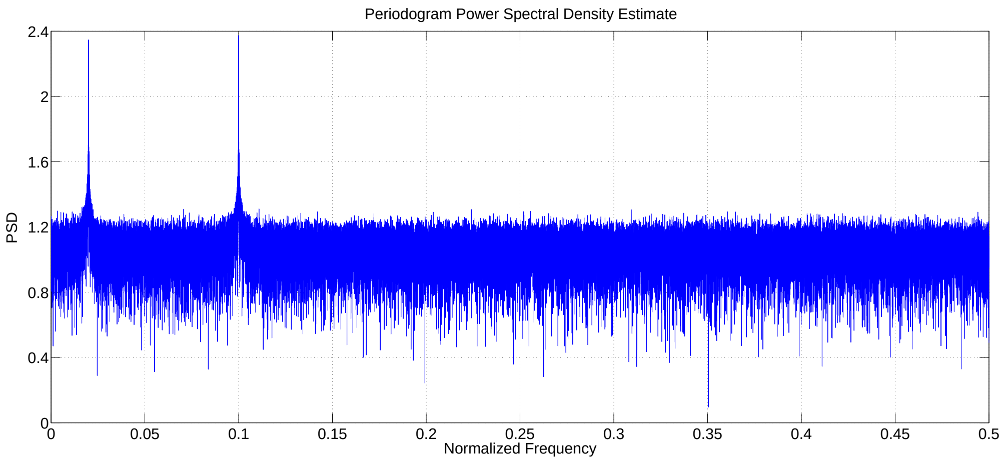
Here comes the core part of the exam .. take time to read the introduction and each problem statement.

Please provide justified, rigorous, and simple answers.

Exercise 2. STREET BAND (20PTS)

During a concert, a street band plays for 5 seconds a combination of tones (each tone is played for 5 seconds), so to provide a signal that can be considered stationary.

We have recorded 100000 samples $[x[1], \ldots, x[100000]]$ (real values) at 20KHz using a microphone that is not really of very good quality. The corresponding periodogram S_X reads



Notice that such a spectrum presents two spikes at 0.02 and 0.1 (normalized frequencies) and a constant baseline.

1) What kind of signal has generated such spectrum? (it can be the sum of different signals)

The spectrum suggests that the street band have played **at least** 2 tones

2) Recalling that the sampling frequency is 20KHz, and that the two spikes are, in normalized frequencies, at 0.02 and 0.1, provide the frequencies in Hz of the two tones.

3) Recalling that we have recorded 100000 samples, is it possible that the street band played more than two tones ? If yes, what are the possible frequencies of the additional tones? if no, why?

We make another recording of the street band. The orchestra plays another combination of tones, but this time, we record only 1 second instead of 5 (each tone is played for 1 seconds). We keep the same sampling frequency as before (20KHz). We do not know exactly how many tones the orchestra has played but we assume that at maximum it has played 5 different tones. We assume that the recording has no noise (now we have used a very good microphone).

4) Propose a parametric method to estimate the frequencies (only the frequencies!) of the played tones and justify you choice.

5) Precisely describe such method: You are given the samples corresponding to a recording of 1 second at 20KHz and you are asked to detail each step as if you have to implement the method in a computer.

6) What happens if the recording is corrupted by noise? Which of the steps that you have precisely described is affected?

Solution 2.

1) The signal that has generated the spectrum is the sum of a two (real) sinusoids (spikes at 0.02 and 0.1) and a white noise (constant baseline). We denote the amplitudes of the two sinusoids with α and β, respectively, and we write a real sinusoid as the sum of two conjugate complex exponentials, that is

$$X[n] = \frac{\alpha}{2}\left(e^{i2\pi n0.02+\Theta_1} + e^{-i2\pi n0.02+\Theta_1}\right) + \frac{\beta}{2}\left(e^{i2\pi n0.1+\Theta_2} + e^{-i2\pi n0.1+\Theta_2}\right) + W[n]\,.$$

Θ_1 and Θ_2 are independent random variables uniformly distributed over $[0, 2\pi]$ that takes the uncertainty of the origin of the sinusoids into account.

2) $f_1 = 0.02 \times 20\text{KHz} = 400\text{Hz}$, and $f_1 = 0.1 \times 20\text{KHz} = 2\text{KHz}$.

3) Yes, it depends on the spectral resolution. Here the spectral resolution is $1/100000$, therefore any tone (spectral line) in the intervals $[0.02-\frac{1}{100000}, 0.02+\frac{1}{100000}]$ and $[0.1-\frac{1}{100000}, 0.1+\frac{1}{100000}]$ will not be visible in the periodogram estimate of the power spectral density.

4) Here the spectral estimation method to be used is the annihilating filter. Indeed, we want to estimate the position of spectral lines (frequencies) and the data is assumed to be noiseless. Any other method will not directly, optimally, and precisely provide the position of the spectral lines.

5) We now have recorded 20000 samples, *i.e.*, $x[1], \ldots, x[20000]$, and we assume that the orchestra has played at most 5 different tones. Given that the recording is real, these 5 tones corresponds to 10 spectral lines within the normalise frequency range 0-1. Therefore, we look for a filter h with 11 coefficients $h[0], h[1], \ldots, h[10]$ (with $h[0] = 1$) such that $(x * h)[n] = 0$. The steps for estimating the frequencies with the annihilating filter approach are:

1) Write $(x * h)[n] = 0$ in matrix form (given $h[0] = 1$)

$$\begin{bmatrix} x[10] & \dots & x[1] \\ x[11] & \dots & x[2] \\ \vdots & & \vdots \\ x[19] & \dots & x[10] \end{bmatrix} \begin{bmatrix} h[1] \\ h[2] \\ \vdots \\ h[10] \end{bmatrix} = - \begin{bmatrix} x[11] \\ x[12] \\ \vdots \\ x[20] \end{bmatrix}$$

obtaining a linear system. Notice that there the samples are $x[1], \dots$ and NOT $x[0], \dots$.

2) Solve the linear system (the matrix is Toeplitz, therefore its solution requires K^2 computations), obtaining $h[1], \dots, h[10]$.

3) Compute the z transform $H[z] = 1 + h[1]z^{-1} + \dots + h[10]z^{-10}$.

4) Find the zeroes of $H[z] = 1 + h[1]z^{-1} + \dots + h[10]z^{-10}$, *i.e.*, the solutions (roots) of the equation $1 + h[1]z^{-1} + \dots + h[10]z^{-10} = 0$. Call $\zeta_1, \dots \zeta_{10}$ the zeroes, giving $H(z) = (1 - \zeta_1 z^{-1}) \dots (1 - \zeta_{10} z^{-1})$.

5) Through the relations $\zeta_1 = e^{i2\pi f_1}, \dots, \zeta_{10} = e^{i2\pi f_{10}}$ we can compute the (normalized) frequencies $f_1, \dots, f_{10}$: The frequency is given by the argument of the complex exponential divided by 2π, that is

$$f_n = \frac{1}{2\pi} \arctan \frac{\mathrm{Im}(\zeta_n)}{\mathrm{Re}(\zeta_n)}$$

(In order to account for numerical errors providing non unitary roots, we can take the argument of $(\zeta_n / |\zeta_n|)$).

6) If the recording is corrupted by noise the zero finding ($1 + h[1]z^{-1} + \dots + h[10]z^{-10} = 0$) becomes a very critical operation, since highly affected by the noise. The 10 roots we obtain using noisy data present a large error when compared to the roots we would have obtained using noiseless data. Consequently, the 10 frequencies we obtain from the 10 roots might be quite different form the true frequencies.

Exercise 3. FILTERED PROCESS (12 PTS)

Let $W[n]$ be a white noise, that is, a i.i.d. sequence of random variables, with variance σ^2 and mean 0. Let $H(z) = 1 - a_1 z^{-1} - a_2 z^{-2}$ be strictly minimum phase filter.

Consider the stochastic process $X[n]$ defined as

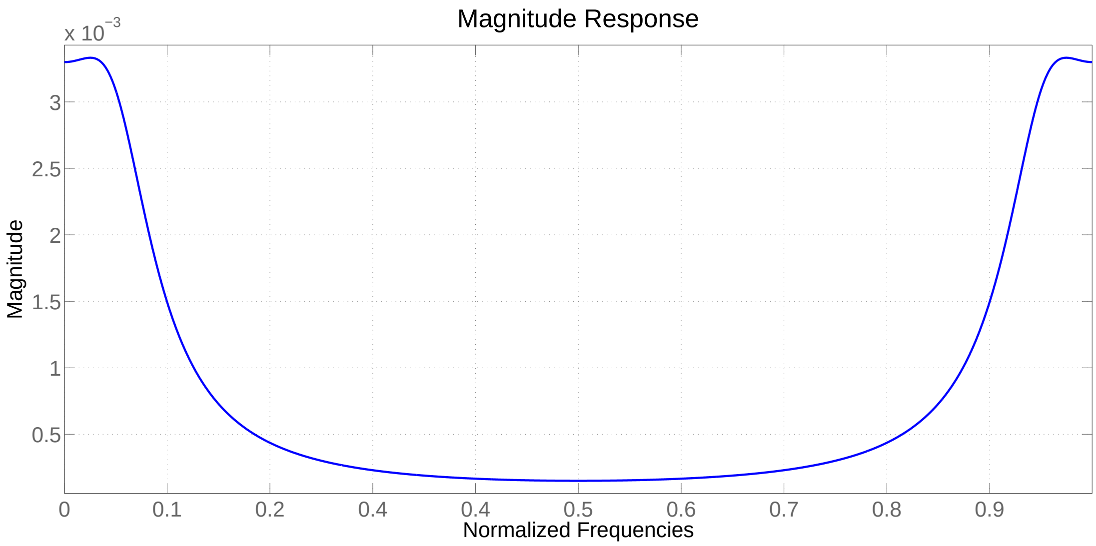
$$X[n]H(z) = W[n] \,,$$

1) Prove that $X[n]$ is a wide-sense stationary process.

2) Compute the expression of its correlation. Justify precisely your answer.

3) Compute the expression of its power spectral density. Justify precisely your answer.

Suppose that the magnitude of the filter $1/H(z)$ is given by



Notice that the maximum value of the magnitude is $3.45\ 10^{-3}$ and the minimum value is $0.15\ 10^{-3}$

4) Draw on the z plane the poles and zeroes of $1/H(z)$.

5) Draw the power spectral density of $X[n]$ and indicate its minimum and maximum values.

Solution 3.

1) $H(z)$ is strictly minimum phase, therefore its inverse $1/H(z)$ defines a stable filter. Consequently, the process $X[n]$ can be modelled as the filtering of a white noise with the stable filter $1/H(z)$. Symbolically, by multiplying on the **right** both terms of the equation by $1/H(z)$, we have

$$X[n] = W[n]\frac{1}{H(z)} \,,$$

Given that a white noise is a w.s.s. process (i.i.d. sequence of random variables), by the fundamental filtering formula the process $X[n]$ is also w.s.s..

2) From its definition $X[n]H(z) = W[n]$, the process $X[n]$ can be written as

$$X[n] = a_1 X[n-1] + a_2 X[n-2] + W[n] \,.$$

The correlation reads

$$\begin{aligned} R(k) =& \mathrm{E}\left[X[n]X^*[n-k]\right] \\ =& \mathrm{E}\left[\left(a_1 X[n-1] + a_2 X[n-2] + W[n]\right) X[n-k]^*\right] \\ =& a_1 R(k-1) + a_2 R(k-2) + \mathrm{E}\left[W[n]X[n-k]^*\right] \,, \end{aligned}$$

where the last term is $\mathrm{E}\left[W[n]X[n-k]^*\right] = \delta_k \sigma^2$.

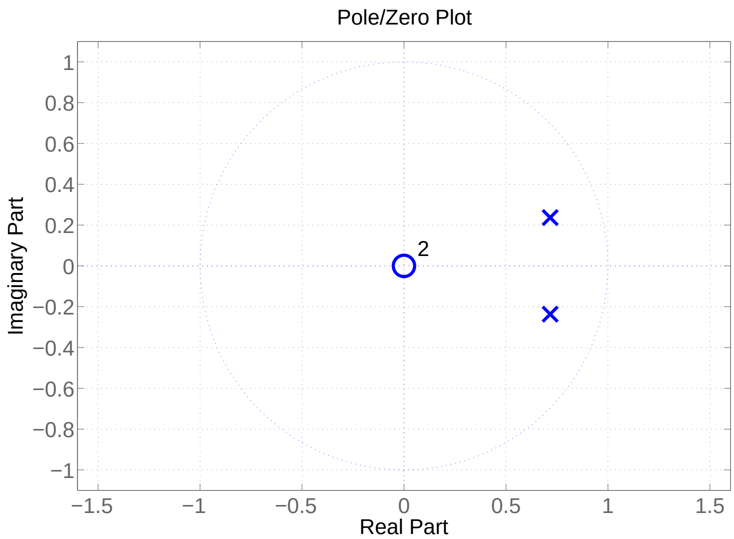
3) We use here the expression of $X[n]$ as the filtering of a white noise

$$X[n] = W[n]\frac{1}{H(z)} \,.$$

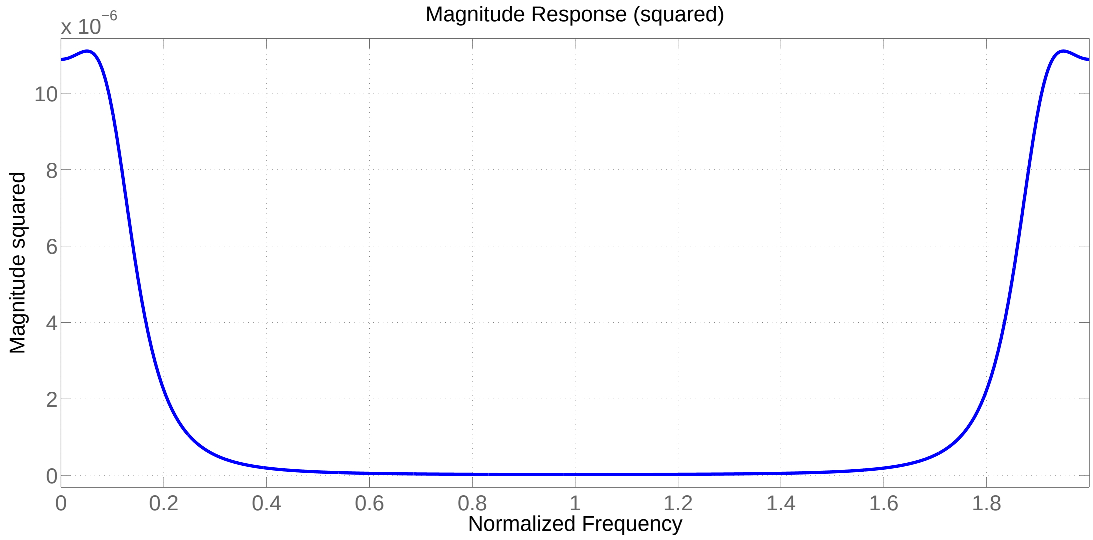
By the fundamental filtering formula (for its applicability see answer 1) we have

$$S_X(\omega) = \frac{1}{|H(e^{i\omega})|^2} S_W(\omega) = \frac{1}{|H(e^{i\omega})|^2}\sigma^2 \,,$$

4) The magnitude plots shows two poles with argument near 0.1 and 0.9 (that is 0.2π and 1.8π). There are no zeroes (behind zeroes at the origin). The z plane plot of the poles reads



5) The power spectral density of $X[n]$ is given by the square of the magnitude of $\frac{1}{H(e^{i\omega})}$ times the variance σ^2 of the noise. That is, the square of the given plot of the magnitude times σ^2.



The maximum value is $\sigma^2(3.45\ 10^{-3})^2 = \sigma^2 11.9\ 10^{-6}$ and the minimum value is $\sigma^2(0.15\ 10^{-3})^2 = \sigma^2 2.25\ 10^{-8}$.