Statistical Signal Processing

Midterm Exam

Thursday, 11 April 2013

You will hand in this sheet together with your solutions.

Write your personal data.

Family Name:

Name:

E-mail:

Read Me First!

Only the personal cheat sheet is allowed.
No class notes, no exercise text or exercise solutions.

Write solutions on separate sheets,
***i.e.* no more than one solution per paper sheet.**

Return your sheets ordered according to problem (solution) numbering.

Return the text of the exam.

A quick question

Exercise 1. How many parameters? (3 pts)
Let $X[n]$, be a stochastic process taking 10 possible values, and such that

- $\mathrm{P}(X[n] = i_0 | X[n-1] = i_1, \ldots, X[n-k] = i_k) = \mathrm{P}(X[n] = i_0 | X[n-1] = i_1)$, for every n, k
- $\mathrm{P}(X[n] = i) = \mathrm{P}(X[n+l] = i)$ for every l

We would like to describe the process $X[n]$ for $n = 1, \ldots, 1000$, that is, we would like to characterize the law (probability) of $X[1], \ldots, X[1000]$. How many parameters are necessary to characterize $X[1], \ldots, X[1000]$?

Solution 1.
$X[n]$ is a Markov chain, characterised by its initial and transition probabilities.
Now, we have 10 possible values (10 states), therefore

- 10 initial probabilities π_i, $i = 1, \ldots, 10$;
- 10×10 transition probabilities p_{ij}, $i = 1, \ldots, 10$, $j = 1, \ldots, 10$.

and

- 1 probability constraint for the initial probabilities $\sum_{i=1}^{10} \pi_i = 1$;
- 10 probability constraints for the transition probabilities $\sum_{j=1}^{10} p_{ij} = 1$, $i = 1, \ldots, 10$.

Finally, the law of the process is characterised by $10 - 1 + 10 \times 10 - 10 = 99$ parameters.
Bonus
If we also take into account that the Markov chain is stationary (*i.e.*, $\mathrm{P}(X[n] = i) = \mathrm{P}(X[n+l] = i)$ for every l), the initial distribution satisfy

$$\pi_j = \sum_{i=1}^{10} \pi_i p_{ij} \,.$$

In such a case the law of the Markov chain is completely characterized by its transition probability, that is, by 90 parameters.

Exercise 2. A AND B (20 PTS)

In a Free Jazz concert two trumpeters play respectively the note A and B for several seconds. The note A can be modelled as a sinusoid with frequency 440Hz, while the note B with a sinusoid of frequency 493.88Hz.

We record the two trumpeters with a sampling frequency of 10KHz.

Modeling

We would like to provide an explicit model, that is, an equation, in order to describe the sound of the two trumpeters.

Given that we do not know the beginning of each sinusoid, we model such an incertitude with a random variable and, therefore, the recorded signal with a stochastic process, that we assume to be w.s.s..

1) Setting the amplitude of the two sinusoids to a_1 and a_2, respectively, provide the expression of the w.s.s. stochastic process $X[n]$ (discrete time) modelling the recorded sound (sampled) produced by the two trumpeter. Clearly describe the random variables modelling the incertitude of the time origin of the sinusoids (distribution, characteristics, etc ..).

2) Prove that the expression you have provided for the stochastic process $X[n]$ describes indeed a w.s.s. process.

Spectral estimation

Let's now go back to the recording, that is, the sequence of samples.

Based on such a sequence of recorded samples we would like to estimate the power spectral density of the process in order to verify that we have indeed recorded the two notes A and B.

We start with the simplest approach: The Periodogram! Obviously, we need to be able to distinguish the two notes, that is, we need an appropriate spectral resolution.

3) How many samples do we need to record in order to be able to distinguish the two notes? Justify precisely your answer.

4) Describe in details the advantages and disadvantages of the Periodogram.

You finally decide not to use the periodogram. Based on the modelling assumption, you are asked to:

5) Propose a parametric spectral estimation method and precisely justify you choice.

6) Describe in detail the proposed spectral estimation method and in particular **ALL** the steps necessary to obtain the values of the spectrum from the recorded samples.

Solution 2.

1) $X[n]$ is a harmonic process, with two harmonics, one at $f_1 = 440Hz$ ($440/10000$ in normalized frequencies) and the other at $f_2 = 493.88$ ($493.88/10000$ in normalized frequencies), where $f_s = 10000$ is the sampling frequency. The easiest way to write the equation of the model is to use complex exponentials, that is

$$X[n] = \frac{a_1}{2}\left(e^{i(2\pi n\frac{f_1}{f_s}+\Theta_1)} + e^{-i(2\pi n\frac{f_1}{f_s}+\Theta_1)}\right) + \frac{a_2}{2}\left(e^{i(2\pi n\frac{f_2}{f_s}+\Theta_2)} + e^{-i(2\pi n\frac{f_2}{f_s}+\Theta_2)}\right).$$

where, given the framework of the problem, the amplitudes a_1 and a_2 are assumed to be real.

Θ_1 and Θ_2 are two random variables modeling the incertitude of the time origin (phase). They are uniformity distributed over $[0, 2\pi)$, and **independent** on each other.

Given that only the random variables Θ_1 and Θ_2 are concerned by the computations of the moments, we shall write the process as

$$\begin{aligned} X[n] =& \frac{a_1}{2}e^{i2\pi n\frac{f_1}{f_s}}e^{i\Theta_1} + \frac{a_1}{2}e^{-i2\pi n\frac{f_1}{f_s}}e^{-i\Theta_1} + \frac{a_2}{2}e^{i2\pi n\frac{f_2}{f_s}}e^{i\Theta_2} + \frac{a_2}{2}e^{-i2\pi n\frac{f_2}{f_s}}e^{-i\Theta_2} \\ =& \alpha[n]e^{i\Theta_1} + \alpha^*[n]e^{-i\Theta_1} + \beta[n]e^{i\Theta_2} + \beta^*[n]e^{-i\Theta_2}, \end{aligned}$$

where obviously $\alpha[n] = \frac{a_1}{2}e^{i2\pi n\frac{f_1}{f_s}}$ and $\beta[n] = e^{i2\pi n\frac{f_2}{f_s}}$.

2) In order to prove that $X[n]$ is w.s.s. we need to verify that

- $\mathrm{E}[X[n]] =$ constant;
- $\mathrm{E}[X[n+k]X[n]^*] = R(k)$, *i.e.*, a function of the time lag difference;
- (Bonus) $\mathrm{Var}X[n] < \infty$, or $\mathrm{E}[|X[n]|] < \infty$.

Now,

- $\mathrm{E}[X[n]] = \mathrm{E}\left[\alpha[n]e^{i\Theta_1} + \alpha^*[n]e^{-i\Theta_1} + \beta[n]e^{i\Theta_2} + \beta^*[n]e^{-i\Theta_2}\right] = \alpha[n]\mathrm{E}\left[e^{i\Theta_1}\right] + \alpha^*[n]\mathrm{E}\left[e^{-i\Theta_1}\right] + \beta[n]\mathrm{E}\left[e^{i\Theta_2}\right] + \beta^*[n]\mathrm{E}\left[e^{-i\Theta_2}\right]$, where

$$\mathrm{E}\left[e^{i\Theta_k}\right] = \int_0^{2\pi} e^{i\theta_k}\frac{1}{2\pi}d\theta = \frac{1}{i2\pi}e^{i\theta_k}\Big]_0^{2\pi} = \frac{1}{i2\pi}(1-1) = 0\,, \quad k = 1, 2\,.$$

- In order to compute $\mathrm{E}[X[n+k]X[n]^*]$, notice that

$$\mathrm{E}\left[e^{\pm i\Theta_1}e^{\pm i\Theta_2}\right] = 0\,,$$

and that

$$\mathrm{E}\left[e^{\pm i2\Theta_k}\right] = 0\,, \quad k = 1, 2\,.$$

Therefore $\mathrm{E}[X[n+k]X[n]^*]$

$$\begin{aligned} =& \mathrm{E}\left[\left(\alpha[n+k]e^{i\Theta_1} + \alpha^*[n+k]e^{-i\Theta_1}\right)\left(\alpha[n]e^{i\Theta_1} + \alpha^*[n]e^{-i\Theta_1}\right)\right] \\ &+ \mathrm{E}\left[\left(\beta[n+k]e^{i\Theta_2} + \beta^*[n+k]e^{-i\Theta_2}\right)\left(\beta[n]e^{i\Theta_2} + \beta^*[n]e^{-i\Theta_2}\right)\right] \\ =& \alpha[n+k]\alpha^*[n] + \alpha[n+k]^*\alpha[n] + \beta[n+k]\beta^*[n] + \beta[n+k]^*\beta[n]\,, \end{aligned}$$

where $\alpha[n+k]\alpha^*[n] = \left(\frac{a_1}{2}\right)^2 e^{i2\pi k\frac{f_1}{f_s}}$ and $\beta[n+k]\beta^*[n] = \left(\frac{a_2}{2}\right)^2 e^{i2\pi k\frac{f_2}{f_s}}$. Without further computation, this shows that the correlation if a function of the difference of the time lags, *i.e.*, $\mathrm{E}[X[n+k]X[n]^*] = R(k)$.

Notice that $R(0) = 2\left(\frac{a_1}{2}\right)^2 + 2\left(\frac{a_2}{2}\right)^2$.

- $\mathrm{Var}\left(X[n]\right) = \mathrm{E}\left[|X[n]|^2\right] - \left|\mathrm{E}\left[X[n]\right]\right|^2 = R(0) - 0 = 2\left(\frac{a_1}{2}\right)^2 + 2\left(\frac{a_2}{2}\right)^2 < \infty\,.$

3) We know that the spectral resolution of a Periodogram is, in normalized frequencies, $1/N$, that is, given N samples we can distinguish two spectral lines is their frequencies differ more than $1/N$. The normalized frequencies of the two sinusoids are f_1/f_s and f_2/f_s. Then

$$\left|\frac{f_2}{f_s} - \frac{f_2}{f_s}\right| = \frac{53.88}{10000} > \frac{1}{N}\,.$$

which gives $N > 185.6$, in practice, $N \geq 186$.

4) Advantages of the Periodogram

   - Simple (just the square of the DFT)
   - General (just assumes that the signal is w.s.s.)

   Disadvantages of the Periodogram

   - Biased
   - Constant variance
   - Affected by the sum of the extreme lag errors of the correlation

5) The process $X[n]$ is harmonic, and its spectrum is composed of spectral lines. Therefore, the optimal approach is to use the annihilating filter. Notice that such a method does not work properly in the presence of noise. However, there is no indication in the presence of noise.

6) Given the assumption that the signal is composed of two (real) sinusoids, with normalized frequencies f_1/f_s and f_2/f_s , the power spectral density presents 4 spectral lines: f_1/f_s, f_2/f_s, with respective amplitudes a_1^2 and a_2^2, and the frequency symmetric counterparts $1 - f_1/f_s$ and $1 - f_2/f_s$, with respective amplitudes a_1^2 and a_2^2. Therefore, we need to estimate 4 frequencies and 2 amplitudes.

   The steps to achieve line spectrum estimation using the annihilating filter are the following

   - Compute the 4 values of the impulse response $h[1], \ldots, h[4]$ ($h[0] = 1$) of the annihilating filter using the following set of equations

$$\begin{bmatrix} x[3] & x[2] & x[1] & x[0] \\ x[4] & x[3] & x[2] & x[1] \\ x[5] & x[4] & x[3] & x[2] \\ x[6] & x[5] & x[4] & x[3] \end{bmatrix} \begin{bmatrix} h[1] \\ h[2] \\ h[3] \\ h[4] \end{bmatrix} = - \begin{bmatrix} x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

   - Compute the the z-transform $H(z) = 1 + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4}$;
   - Find the zeroes of $H(z)$ (with respect to the variable z);

- Given that the zeroes have the form $z_k = e^{i2\pi\frac{f_k}{f_s}}$, k=1,2,3,4, (where $f_3/f_s = 1 - f_2/f_s$ and $f_4/f_s = 1 - f_1/f_s$), compute the frequencies f_k, k=1,2,3,4, as the scaled argument of the zeroes

$$f_k = \frac{f_s}{2\pi}\arg\left(z_k\right) .$$

- Given the line spectrum positions f_k, $k = 1, 2, 3, 4$, compute the amplitudes of the line spectrum using the equation

$$X[n] = \frac{a_1}{2}e^{i2\pi n\frac{f_1}{f_s}}e^{i\Theta_1} + \frac{a_1}{2}e^{-i2\pi n\frac{f_1}{f_s}}e^{-i\Theta_1} + \frac{a_2}{2}e^{i2\pi n\frac{f_2}{f_s}}e^{i\Theta_2} + \frac{a_2}{2}e^{-i2\pi n\frac{f_2}{f_s}}e^{-i\Theta_2} ,$$

that is, solve the linear system

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ e^{i2\pi\frac{f_1}{f_s}} & e^{-i2\pi\frac{f_1}{f_s}} & e^{i2\pi\frac{f_2}{f_s}} & e^{-i2\pi\frac{f_2}{f_s}} \\ e^{i4\pi\frac{f_1}{f_s}} & e^{-i4\pi\frac{f_1}{f_s}} & e^{i4\pi\frac{f_2}{f_s}} & e^{-i4\pi\frac{f_2}{f_s}} \\ e^{i6\pi\frac{f_1}{f_s}} & e^{-i6\pi\frac{f_1}{f_s}} & e^{i6\pi\frac{f_2}{f_s}} & e^{-i6\pi\frac{f_2}{f_s}} \end{bmatrix} \begin{bmatrix} \frac{a_1}{2}e^{j\Theta_1} \\ \frac{a_1}{2}e^{-j\Theta_1} \\ \frac{a_2}{2}e^{j\Theta_2} \\ \frac{a_2}{2}e^{-j\Theta_2} \end{bmatrix} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[4] \end{bmatrix}$$

Then the amplitudes of the spectral lines are given by the **square of the absolute value** of the solution of the above linear system, *i.e.*, $|\frac{a_1}{2}e^{j\Theta_1}|^2$, $|\frac{a_1}{2}e^{-j\Theta_1}|^2$, $|\frac{a_2}{2}e^{j\Theta_2}|^2$, $|\frac{a_2}{2}e^{-j\Theta_2}|^2$.

Exercise 3. FILTERED PROCESS (15 PTS)

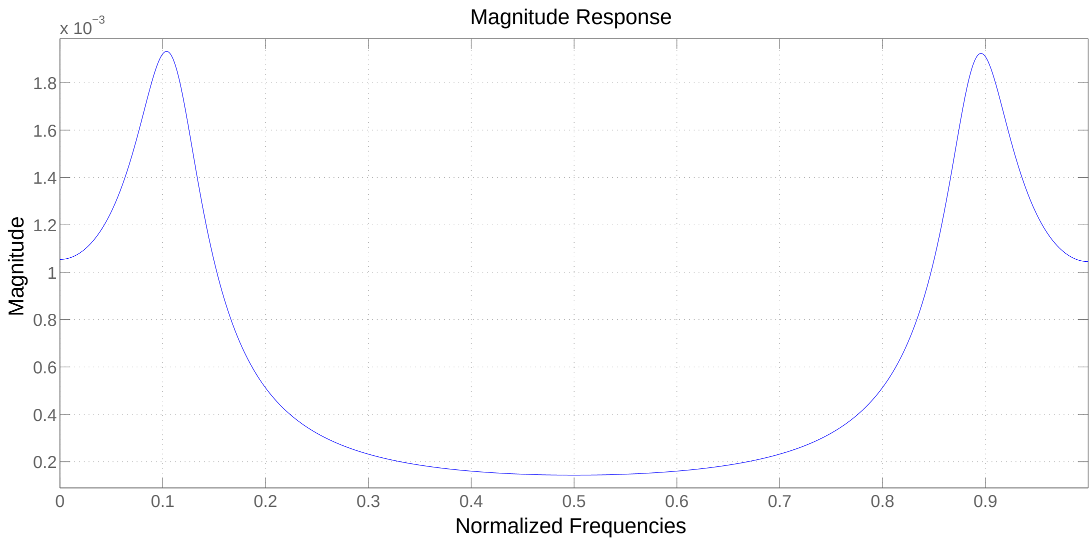
Let $W[n]$ be a white noise, that is, a i.i.d. sequence of random variables, with variance σ^2 and mean 0. Let $H(z) = 1 - a_1 z^{-1} - a_2 z^{-2}$ be strictly minimum phase filter.

Consider the stochastic process $X[n]$ defined as

$$X[n]H(z) = W[n]\,,$$

1) Prove that $X[n]$ is a wide-sense stationary process.
2) Compute the expression of its correlation. Justify precisely your answer.
3) Compute the expression of its power spectral density. Justify precisely your answer.

Suppose that the magnitude of the filter $1/H(z)$ is given by



Notice that the maximum value of the magnitude is 1.95 10^{-3} and the minimum value is 0.15 10^{-3}

4) Draw on the z plane the poles and zeroes of $1/H(z)$.
5) Draw the power spectral density of $X[n]$ and indicate its minimum and maximum values.

Solution 3.

1) $H(z)$ is minimum phase, therefore its inverse $1/H(z)$ defines a stable filter. Consequently, the process $X[n]$ can be modeled as the filtering of a white noise with the stable filter $1/H(z)$. Symbolically, by multiplying on the right both terms of the equation by $1/H(z)$, we have

$$X[n] = W[n]\frac{1}{H(z)}\,,$$

Given that a white noise is a w.s.s. process (i.i.d. sequence of random variables), by the fundamental filtering formula the process $X[n]$ is also w.s.s..

2) From its definition $X[n]H(z) = W[n]$, the process $X[n]$ can be written as

$$X[n] = a_1 X[n-1] + a_2 X[n-2] + W[n]\,.$$

The correlation reads

$$\begin{aligned} R(k) =& \mathrm{E}\left[X[n+k]X^*[n]\right] \\ =& \mathrm{E}\left[X[n+k]\left(a_1 X[n-1] + a_2 X[n-2] + W[n]\right)^*\right] \\ =& a_1 R(k+1) + a_2 R(k+2) + \mathrm{E}\left[X[n+k]W^*[n]\right]\,, \end{aligned}$$

where the last term is $\mathrm{E}\left[X[n+k]W^*[n]\right] = \delta_k \sigma^2$.

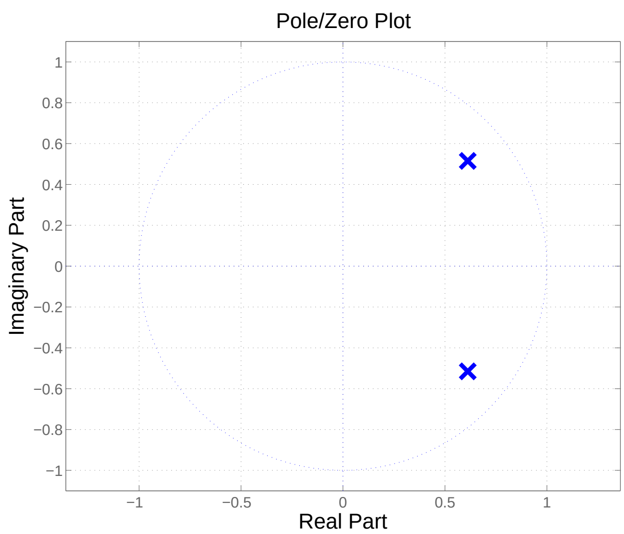
3) We use here the expression of $X[n]$ as the filtering of a white noise

$$X[n] = W[n]\frac{1}{H(z)}\,.$$

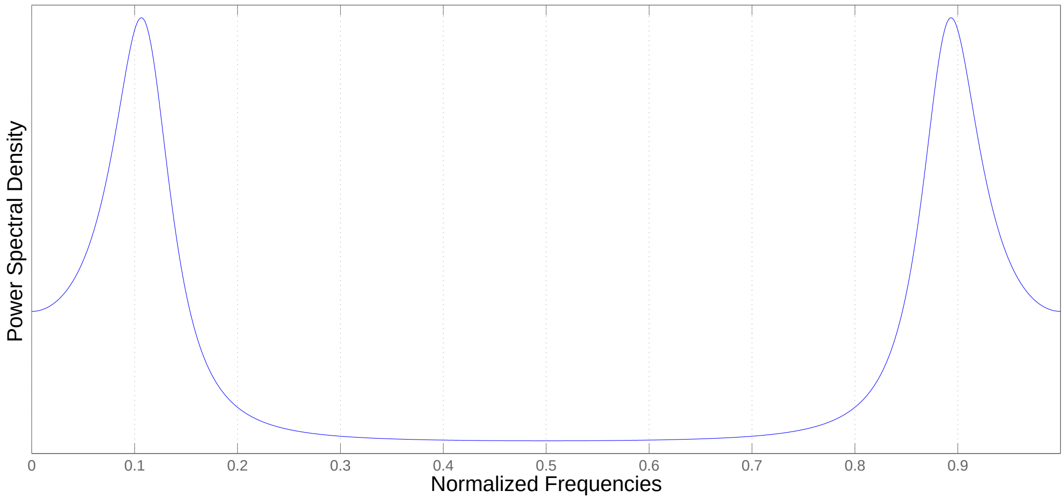
By the fundamental filtering formula (for its applicability see answer 1) we have

$$S_X(\omega) = \frac{1}{|H(e^{i\omega})|^2} S_W(\omega) = \frac{1}{|H(e^{i\omega})|^2}\sigma^2\,,$$

4) The magnitude plots shows two poles with argument near 0.1 and 0.9 (that is 0.2π and 1.8π). There are no zeroes (behind zeroes at the origin). The z plane plot of the poles reads



5) The power spectral density of $X[n]$ is given by the square of the magnitude of $\frac{1}{H(e^{i\omega}}$ times the variance σ^2 of the noise. That is, the square of the given plot of the magnitude times σ^2.



The maximum value is $\sigma^2(1.95\ 10^{-3})^2 = \sigma^2 3.8\ 10^{-6}$ and the minimum value is $\sigma^2(0.15\ 10^{-3})^2 = \sigma^2 2.25\ 10^{-8}$.