Exercises 9

Exercise 1. Correlating and Decorrelating Signals
In this exercise, we will see that a signal can be both correlated and decorrelated by applying a suitable linear transform, where in the latter case the optimal transform is the Karhunen–Loeve transform (KLT). KLT in signal processing litterature is basically equivalent to PCA.

(a) Consider an i.i.d. sequence of random variables $X_0, X_1, \ldots, X_{N-1}$ ($\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] = 1$, for $i = 0, 1, \ldots, N-1$). Define a new set of random variables $\mathbf{Y} = [Y_0, Y_1, \ldots, Y_{N-1}]^T$ as

$$\mathbf{Y} = \mathbf{A} \cdot \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix},$$

where

$$\mathbf{A} = \begin{bmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,N-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,N-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{N-1,0} & \alpha_{N-1,1} & \cdots & \alpha_{N-1,N-1} \end{bmatrix}$$

is a real square matrix.

Show that the correlation function satisfies:

$$R_{i,j} = E[Y_i \cdot Y_j] = \sum_{k=0}^{N-1} \alpha_{i,k} \cdot \alpha_{j,k},$$

for $i, j = 0, 1, \ldots, N-1$.

(b) Show that the following equality holds:

$$\det(\mathbf{A}) = \prod_{i=0}^{N-1} \lambda_i^{1/2},$$

where the λ_i are eigenvalues of the correlation matrix $\mathbf{R}_y$.

(c) Consider a time sequence of random vectors $\mathbf{Y}[n] = [Y_0[n], Y_1[n], \ldots, Y_{N-1}[n]]^T$. The KLT of the random signal $\mathbf{Y}[n]$ is obtained as $\mathbf{Z}[n] = \mathbf{T} \cdot \mathbf{Y}[n]$, where the rows of the matrix $\mathbf{T}$ are the eigenvectors of the correlation matrix of the signal $\mathbf{Y}[n]$ (sorted in descending order of the corresponding eigenvalues).

Show that the resulting vector coefficients $\mathbf{Z}[n]$ are uncorrelated. Are they independent?

Exercise 2. KLT OF CIRCULANT CORRELATION MATRICES

Let X be a real periodic sequence of period $N = 4$ with correlation matrix R_x:

$$R_x = \begin{bmatrix} 1 & 0.4 & 0.2 & 0.4 \\ 0.4 & 1 & 0.4 & 0.2 \\ 0.2 & 0.4 & 1 & 0.4 \\ 0.4 & 0.2 & 0.4 & 1 \end{bmatrix}$$

(a) Compute its KLT, that is, the transform T that diagonalizes R_x.

(b) Consider now the DFT matrix S_N of size $N = 4$. Compute $S_N^* R_x S_N$. What do you obtain? Recall that the DFT can be formulated as a complex matrix multiplication $X[k] = S_N x[n]$ where the DFT matrix S_N is given by $S_N[k,n] = W_N^{-kn}$.

(c) Compare both solutions. What can you conclude?

Exercise 3. AUTOMATIC CLASSIFICATION OF SOUND WAVES

A brilliant engineer has developed an automatic detection system that records sound activity of dolphins and classifies them among 4 characteristic sound waves, that we will call α, β, γ, δ. The system is installed on a boat and performs detection on the fly. A radio subsequently transmits to a base station the wave category (α, β, γ, or δ) coded as numerical values. Let $X[n]$ denote the obtained physical signal, and we assume $X[n]$ is a Markov chain (with 4 values corresponding to the 4 states α, β, γ, and δ).

While the detection system is brilliant, the transmission one is not as good. Indeed, the numerical values coding the 4 types of waves are not known at the receiver, and the received signal $Y[n]$ is quite noisy (additive Gaussian white noise). This makes the decoding process difficult.

Consequently, at the receiver, we first need to denoise the signal $Y[n]$ to estimate $X[n]$ prior to any decoding.

- Given that we have observed $N = 1000$ samples of the noisy signal, *i.e.*, $y[1], \ldots, y[1000]$, propose a denoising method, justifying your choice. Describe the method in detail, clearly writing the equations of the models used. Write every step of your method (as a bullet list) clearly indicating the input and output of each step. The final output of the question is an estimation of $x[1], \ldots, x[1000]$.