Exercises 6

Exercise 1. ANNIHILATING FILTER METHOD VS. MUSIC
Assume we have a random process $X[n]$ composed of 3 complex sinusoids:

$$X[n] = \sum_{k=1}^{3} \alpha_k e^{j(2\pi f_k n + \Theta_k)},$$

where $(f_1, f_2, f_3) = (0.2, 0.3, 0.4)$, $(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 3)$ and the phases Θ_k are stationary random variables, independent and uniformly distributed over $[0, 2\pi)$. The signal is affected by additive zero-mean white noise of variance σ_W^2, independent of $X[n]$. We only have access to the noisy realizations, i.e.

$$Y[n] = X[n] + W[n].$$

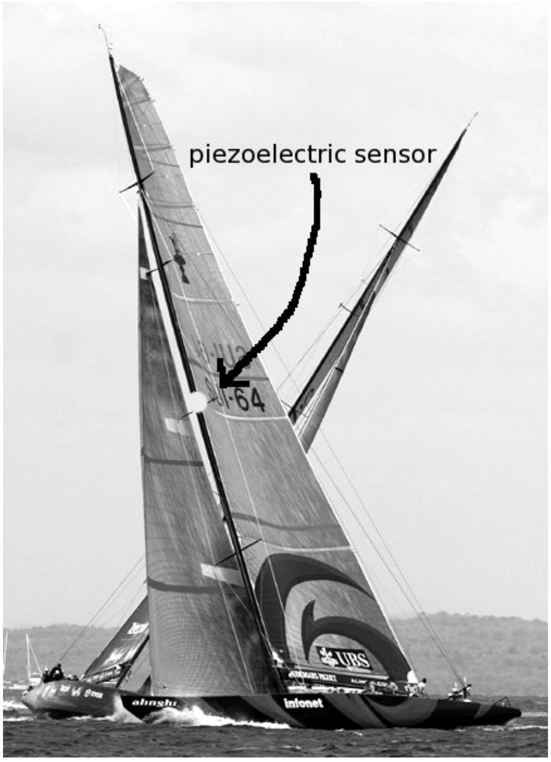
(a) Simulate 20 realizations of $Y[n]$ when $\sigma_W^2 = 1$ and estimate the frequencies f_k and weights α_k of the sinusoids using:
  (a) annihilating filter method,
  (b) MUSIC method.

(b) Do the same steps when $\sigma_W^2 = 4$ and compare the two methods.

(c) Assume the signal $X[n]$ is deterministic, i.e. the phases Θ_k are known. We want to estimate f_k and α_k. Can we now use the annihilating filter method and the MUSIC method? Point out the differences.

Exercise 2. ALINGHI II: MAST STRESS ANALYSIS (*16 points*)
The Mast (the large pole used to hold up the sails) is definitively a critical component of the sailing boat. Here again, the high technology materials used for its construction are pushed to their stress limit. During prototype testings, the behavior of the Mast must be monitored so to assure that it is properly dimensioned: if the mast breaks, the game is over (as for NZ team in the 2003 edition).

More precisely, we monitor the elongations of the Mast using a piezoelectric sensor positioned at its middle point, as depicted in the figure below.



Much of the information on the mast stress is contained in the power spectrum of the signal measured by the piezoelectric sensor. In particular such a power spectrum is smooth and can be approximated by a fractional polynomial

$$S(\omega) = \left. \frac{1}{C(z)} \right|_{z=e^{j\omega}}$$

1) Assuming that the approximation of a smooth spectrum (fractional polynomial) is correct, precisely describe a method for estimating the spectrum. More precisely we need to

   1.a) Estimate the number of parameters describing the spectrum (order of the polynomial, etc.)

   1.b) Estimate the value of such parameters

   1.c) Provide an estimation error

We then realize that the smooth spectrum (fractional polynomial) approximation is not exactly correct.

2) How this will affect the estimation of the number and values of the parameters?

The technical team complains that the method you have proposed is too complicated and ask you to use a periodogram based approach

3) Give precise arguments to defend your choice.

Exercise 3. PRIOR, POSTERIOR, LIKELIHOOD
Suppose the signal $X[n]$ is constant, with amplitude A. Furthermore, let the observed signal $Y[n]$ contain a noisy version of the signal $X[n]$

$$Y[n] = X[n] + W[n]\,,$$

where $W[n]$ is a zero-mean white Gaussian noise with variance $\sigma^2 = 1$.

(a) Assuming N observations of $Y[n]$ are given $(Y[0], \ldots, Y[N-1])$, give the *maximum likelihood estimation* of amplitude A

(b) Download data file `Review_3.csv` from Moodle. It was generated from Y. Estimate θ *using the formulas you derived in the previous part.*

(c) Now assume that A is in itself a random variable, with prior distribution $A \sim \mathcal{N}(0, \sigma_A^2)$. What is MAP (maximum a posteriori) estimator of A?

(d) How can you relate *prior, posterior* and *likelihood* in one equation?

(e) Assuming different σ_A^2 (for example $0.001, 1$ and 100), estimate from the data A using MAP. What do you observe?