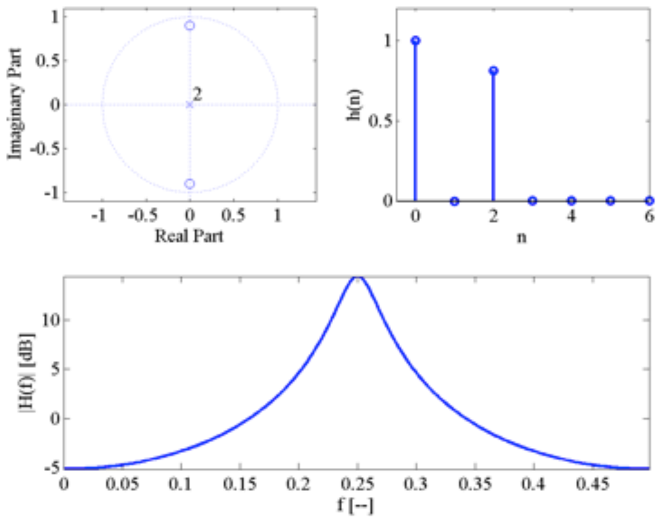
Exercises 2

Exercise 1. One system or more than one system?
Considering the following 3 plots:

- A z-transform in the z-plane (upper left corner), where `o` denotes the zeros and `x` the poles (the poles can in this framework be neglected);
- An impulse response $h(n)$ in the time domain (upper right corner);
- Magnitude of the frequency response $|H(e^{j2\pi f})|$ in normalized frequencies (bottom).



According to the plots:

(a) Do the z-transform and the impulse response $h(n)$ correspond to the same system (that is, is the plot of the z-plane the plot of the z-transform of $h(n)$)? You can plot the z-transform in Matlab and compare the plots.

(b) Do the z-transform and the magnitude of the frequency response $|H(e^{j2\pi f})|$ correspond to the same system (that is, is $|H(e^{j2\pi f})|$ the absolute value on the unit circle of the z-transform represented in the z-plane plot)?

Exercise 2. Hilbert Spaces in Probability.
Consider the random variables X_0, X_1, X_2 defined on the same probability space. Suppose that

the mean of each variable is 0 and the joint correlation matrix is

$$\mathbf{R_X} = \mathbb{E}[[X_0 X_1 X_2]^T [X_0 X_1 X_2]] = \begin{bmatrix} 8 & 4 & 1 \\ 4 & 8 & 4 \\ 1 & 4 & 8 \end{bmatrix}.$$

Let us define Hilbert space H as the space generated by all the linear combinations of the variables X_0, X_1, and X_2, i.e.

$$H = \{a_0 X_0 + a_1 X_1 + a_2 X_2, a_0, a_1, a_2 \in \mathbb{R}\}.$$

(a) Determine an orthogonal basis, $\{Y_0, Y_1\}$ for the subspace W generated by X_0 and X_1.

(b) Find the best approximation of the variable X_2 in the subspace W, i.e. the random variable Y that minimizes $\mathbb{E}[|Y - X_2|^2]$, with $Y \in W$. (Hint: apply the projection theorem.)

Exercise 3. Links between definitions

In this exercise we try to tackle what definitions imply other definitions. For each statement below, show if it is always true or not. (If it is false, a counter-example is sufficient.)

(a) The Power Spectral Density of a real-valued process is also real-valued.

(b) If a stochastic process is SSS, then the random variables in the process are i.i.d.

(c) If two random variables are independent, they are uncorrelated.

Exercise 4. A Simple AR Process

Consider the discrete time stochastic process $\{X[n]\}_{n\geq 0}$ defined by

$$X[n+1] = aX[n] + W[n+1], \quad n \geq 0$$

where $|a| < 1$, $X[0]$ is a Gaussian random variable of mean 0 and variance c^2, and $\{W[n]\}_{n\geq 1}$ is a sequence of i.i.d. Gaussian variables of mean 0 and variance σ^2, and independent of $X[0]$.

(a) Express $X[n]$ in terms of $X[0], W[1], \ldots, W[n]$ (and a). Give the mean and variance of $X[n]$;

(b) Suppose now that $c^2 = \frac{\sigma^2}{1-|a|^2}$. Show that with this specific choice for the variance of $X[0]$, the process $\{X[n]\}_{n\geq 0}$ is strictly stationary.

(c) Give the one-step predictor of $X[n]$: $\widehat{X}[n|n-1]$.

(d) What is the whitening (or analysis) filter of $\{X[n]\}_{n\in\mathbb{Z}}$?
What is the generating (or synthesis) filter of $\{X[n]\}_{n\in\mathbb{Z}}$?

(e) Give the covariance function $R_X[k] = \mathbb{E}\left[X[n+k]X[n]^*\right]$.

(f) Write $X[n]$ in terms of $W[n]$, $W[n-1]$ and $X[n-2]$. Deduce from this the two-step predictor of $X[n]$: $\widehat{X}[n|n-2]$, the projection of $X[n]$ onto $H(X, n-2)$, the Hilbert subspace spanned by the random variables $X[n-2], X[n-3], \ldots$.