Statistical Signal Processing

Final Exam

Wednesday, 18 June 2014

Write your personal data (please make it readable!)

Family Name:

Name:

EPFL email:

Read Me First!

- Only pens (no pencils), the personal cheat sheet, and a calculator are allowed (blank paper sheets for writing solution are provided)
- Write solutions on separate sheets, *i.e.* no more than one solution per sheet
- Return your sheets ordered according to problem (solution) numbering
- Return this page and the text of the exam
- Read carefully each question before answering

Warmup exercises

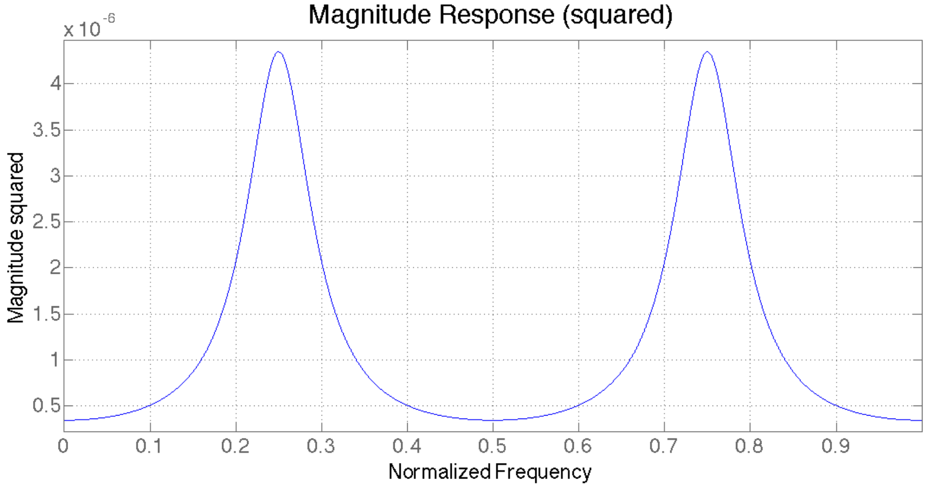
These are warm up problems .. do not spend too much time on them.

Please provide justified, rigorous, and simple answers.

Exercise 1. A FILTER (6PTS)

Consider a linear time-invariant filter.

The plot below depicts the square magnitude of its transmittance $H(e^{j\omega})$ that has been computed as the square module of the discrete time Fourier transform of the impulse response $h[n]$.



Based on the given square magnitude is the filter

1) stable or instable?

2) causal or anti-causal?

Please justify precisely each answer.

3) Draw poles and zeros accordingly.

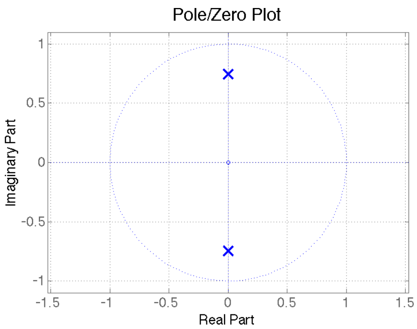
Solution 1.

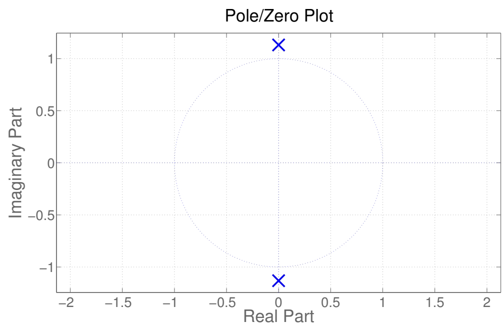
1) The transmittance has been computed as the DTFT of the impulse response $h[n]$. In order to be defined (to exist), the DTFT requires that $h[n] \in \ell^1$. Therefore the filter is stable.

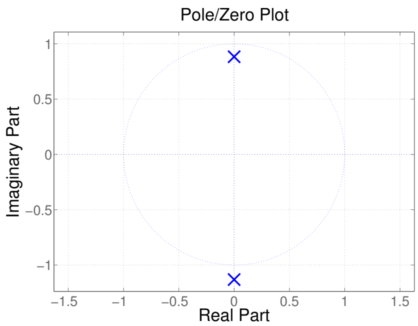
2) The filter can be

   – Causal, with poles inside the unit cercle
   – Anti-causal, with poles outside the unit circle
   – Non-causal (sum of a causal part and an anti-causal part), with poles inside the unit cercle for the causal part and poles outside for the anti-causal part.

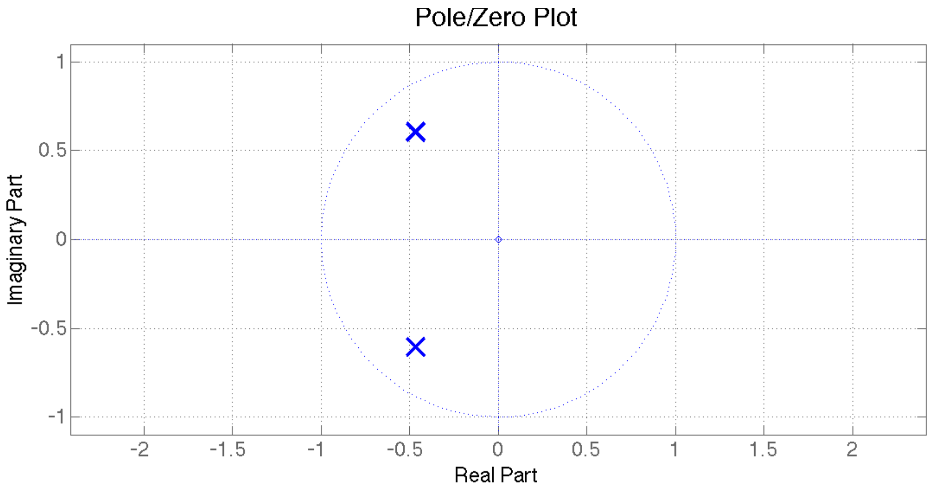
3) If we consider, respectively, the causal, the anti-causal, and the non causal cases, the poles are as follows







Exercise 2. YEAT ANOTHER FILTER (6PTS)
We now have **another** filter, with two poles, as depicted in the plot below.



Based on the above zero-pole plot, is the filter

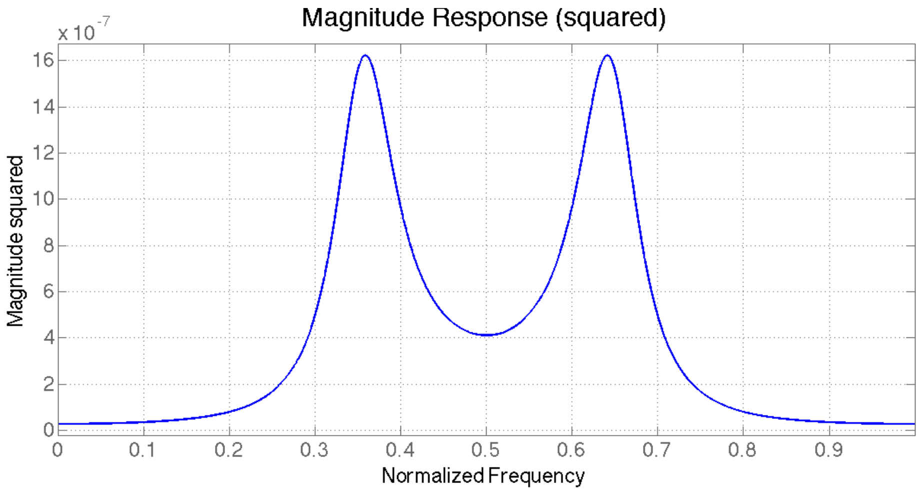
1) stable or instable?
2) causal or anti-causal?

Please justify precisely each answer.

3) If possible, draw the magnitude (or the square magnitude) of the filter in the frequency domain, accordingly.

Solution 2.

1) if considered causal, the filter is stable. If considered anti-causal, the filter is unstable
2) can be both
3) Considering the filter stable and causal, the square magnitude of the transmittance evaluated on the unit cercle reads



If the filter is considered anti-causal, then it is not stable and the transmittance evaluated on the unit cercle (DTFT of the impulse response) does not exist.

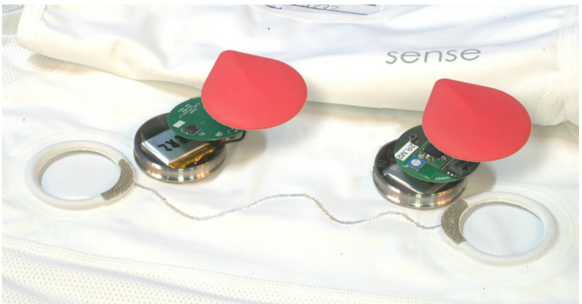
Main exercises

Here comes the core part of the exam .. take time to read the introduction and each problem statement.

Please provide justified, rigorous, and simple answers.

Exercise 3. A BIOMEDICAL DEVICE AND ITS INTERFERENCES (25 PTS)

A team of engineers works at the development of a portable biomedical system targeting physiological measurements such as ECG, respiration rate, blood oxygenation, core body temperature, and skin ph level, for top level athletes. Such a system has an internal clock fixing the sampling frequency $f_s = 1$KHz.



While testing the electronics for ECG measurements, they discover that such measurement are highly affected by interferences. We shall call in the following $X[n]$ the ECG signal (without interferences), and $Y[n]$ the interference signal (over short time segments, they are both modeled as w.s.s. processes).

They have been able to determine that such interferences are due to control signals generated by the electronics for core body temperature measurements. Their theoretical computations of such interference leads to a signal of the type

$$S[n] = \alpha \times cos(\omega_1 n + \Theta) \times cos(\omega_2 n)\,, \quad \Theta \sim U([0, 2\pi])\,,$$

where they think $\omega_1 = 2\pi \times 0.01$, and $\omega_2 = 2\pi \times 0.1$.

However, they are not sure whether there is also an additional white noise, that is $Y[n] = S[n] + W[n]$.

They decide to use the periodogram, first of all, to validate the theoretically computed interference $S[n]$, and secondly, to check whether there is also a white noise.

1) Given the expression of $S[n]$, how many samples $y[n]$ they need to measure so to be able to validate such theretical model? Justify precisely your answer.

The periodogram shows that there is no white noise, that is $Y[n] = S[n]$.

2) Given the expression of $S[n]$ (and therefore of $Y[n]$), sketch its periodogram (computed using the number of samples you have given in 1)) over the normalized frequency interval [0,1].

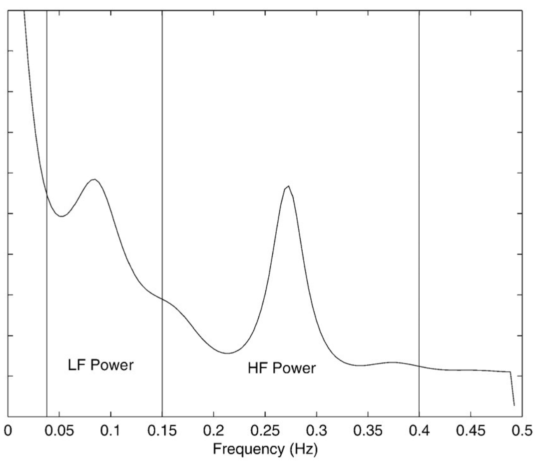
We would like to precisely estimate the frequencies of the two cosines, ω_1 and ω_2, and the amplitude α (the periodogram is too bad for this).

3) Propose a parametric spectral estimation method than enables to estimate ω_1, ω_2, and α. Justify precisely your answer.

4) Given that we have measured $y[1], \ldots, y[10000]$, describe in detail the method, step by step, from the measured samples to the estimated spectrum, like if each step has to be interpreted and executed by a computer (in particular the input, the executed operation with corresponding equations, and the output of each step has to be clear)

Thanks to your contribution the interferences have been deeply analyzed and eliminated. We now have access to a noise-free ECG signal $X[n]$.

One of the goal of measuring an athlete's ECG is to compute the intervals between each heart beat (called RR signal) and to analyze the corresponding spectrum (that is, the spectrum of the RR signal). In particular we want to analyze the spectral power (integral of the spectrum) over two frequency intervals. A typical spectrum of the RR signal is depicted in the figure below, with the frequency interval of interest, namely LF and HF.



Call $RR[n]$ the RR signal and $rr[1], \ldots, rr[10000]$ the samples obtained via the ECG measurement.

5) Propose a parametric spectral estimation method to estimate the spectrum of the RR signal. Justify precisely your answer.

6) Given that we have measured $rr[1], \dots, rr[10000]$, describe in detail the method, step by step, from the measured samples to the estimated spectrum, like if each step has to be interpreted and executed by a computer (in particular the input, the executed operation with corresponding equations, and the output of each step has to be clear)

Solution 3.

1) A quick computation shows that the signal $S[n]$ is not w.s.s.. Indeed

$$\begin{aligned}
\mathrm{E}\left[S[n+k]S[n]\right] &= \alpha^2 \cos(\omega_2(n+k))\cos(\omega_2 n)\mathrm{E}\left[\cos(\omega_1(n+k)+\Theta)\cos(\omega_1 n+\Theta)\right] \\
&= \frac{\alpha^2}{4}\left(\cos(\omega_2 k) + \cos(\omega_2(2n+k))\right)\mathrm{E}\left[\cos(\omega_1 k) + \cos(\omega_1(2n+k)+2\Theta)\right] \\
&= \frac{\alpha^2}{4}\left(\cos(\omega_2 k) + \cos(\omega_2(2n+k))\right)\cos(\omega_1 k)
\end{aligned}$$

Within such a framework, the periodogram is to be simply interpreted as square of the Fourier transformation of the measured samples.

The signal $S[n]$ takes the form

$$\begin{aligned}
S[n] =&\alpha \times cos(\omega_1 n + \Theta) \times cos(\omega_2 n) \\
=&\alpha\frac{1}{4}\left(e^{i(\omega_1 n+\Theta)} + e^{-i(\omega_1 n+\Theta)}\right)\left(e^{i(\omega_2 n)} + e^{-i(\omega_2 n)}\right) \\
=&\alpha\frac{1}{4}\left(e^{i((\omega_1+\omega_2)n+\Theta)} + e^{-i((\omega_1+\omega_2)n+\Theta)} + e^{i((\omega_1-\omega_2)n+\Theta)} + e^{-i((\omega_1-\omega_2)n+\Theta)}\right) . \qquad (1)
\end{aligned}$$

Θ does not affect the frequency content of the signal (it does not depends on the time index!), but only the phase of the signal. Consequently it has no effect on the module of the Fourier transformation. Therefore the frequency content of the signal is composed of 4 spectral lines, at frequencies (radiants) $-(\omega_1+\omega_2) = -2\pi 0.11, (\omega_1-\omega_2) = -2\pi 0.09, -(\omega_1-\omega_2) = 2\pi 0.09, (\omega_1+\omega_2) = 2\pi 0.11$. Notice that such a result can also be straightforwardly obtained using the multiplication-convolution property of the Fourier transformation.

The smallest frequency interval between the 4 spectral lines is $2\omega_1 = 4\pi 0.01$ (radiants). Consequently, in order to be able to distinguish the spectral lines when using the periodogram, we need to take at least

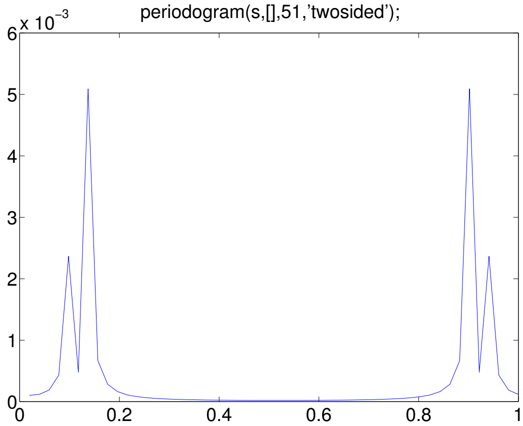
$$N > \frac{2\pi}{2\omega_1} = \frac{2\pi}{4\pi 0.01} = \frac{1}{0.02} = 50$$

samples, that is, at least 51 samples.

2) We have 4 line spectra and a periodogram computed using the number of sample computed in the previous question, that is 51. Therefore the key characteristic of the sketch is: 4 spectral line that, by pairs, can barely be distinguished.

   Just to give an example, the figure below shows the matlab periodogram computed using 51 samples, where

```
>> n=1:51;
>> a=0.1;
>> w2=2*pi*0.1;
>> w1=2*pi*0.01;
>> s=a*cos(w1*n).*cos(w2*n);
>> PS=periodogram(s,[],51,'twosided');
>> x=(1:51)/51;
>> figure; plot(x,PS)
```



   To be noticed that, being at the limit of spectral resolution, the line spectra do not appear to have the same amplitude!

3) The signal $Y[n] = S[n]$ is clearly an harmonic signal (see equation (1)) with a spectrum characterized by 4 spectral lines, and in the total absence of noise. Therefore the annihilating filter will enable to estimate its frequencies and the absolute value of the amplitude. Notice that α can be assumed to be positive (a change of sign can taken into account by Θ). The annihilating filter approach will enable to estimate $\omega_1 + \omega_2$, $\omega_1 - \omega_2$, $-\omega_1 - \omega_2$, and $-\omega_1 + \omega_2$ from which we can easily compute ω_1 and ω_2, and then $|\alpha| = \alpha$ (when assumed to be positive.

4) We have recorded $y[1], \ldots, y[10000]$ and we need to estimate 4 spectral lines. Therefore we look for a filter with 5 coefficients $h[0], h[1], \ldots, h[4]$ (with $h[0] = 1$) such that $(y * h)[n] = 0$. The steps for estimating the frequencies with the annihilating filter approach are:

1) Write $(y * h)[n] = 0$ in matrix form (given $h[0] = 1$)

$$\begin{bmatrix} y[4] & \dots & y[1] \\ y[5] & \dots & y[2] \\ \vdots & & \vdots \\ y[7] & \dots & y[4] \end{bmatrix} \begin{bmatrix} h[1] \\ h[2] \\ \vdots \\ h[4] \end{bmatrix} = - \begin{bmatrix} y[5] \\ y[6] \\ \vdots \\ y[8] \end{bmatrix}$$

obtaining a linear system. Notice that the samples are $y[1], \dots$ and NOT $y[0], \dots$.

2) Solve the linear system (the matrix is Toeplitz, therefore its solution requires 4^2 computations), obtaining $h[1], \dots, h[4]$.

3) Compute the z transform $H[z] = 1 + h[1]z^{-1} + \dots + h[4]z^{-4}$.

4) Find the zeroes of $H[z] = 1 + h[1]z^{-1} + \dots + h[4]z^{-4}$, *i.e.*, the solutions (roots) of the equation $1 + h[1]z^{-1} + \dots + h[4]z^{-4} = 0$. Call $\zeta_1, \dots, \zeta_4$ the zeroes, giving $H(z) = (1 - \zeta_1 z^{-1}) \dots (1 - \zeta_4 z^{-1})$.

5) Through the relations $\zeta_1 = e^{i\widetilde{\omega}_1}, \dots, \zeta_4 = e^{i\widetilde{\omega}_4}$ we can compute the frequencies $\widetilde{\omega}_1, \dots, \widetilde{\omega}_4$ (in radiants): The frequency is given by the argument of the complex exponential, that is

$$\widetilde{\omega}_n = \arctan \frac{\mathrm{Im}(\zeta_n)}{\mathrm{Re}(\zeta_n)}$$

(In order to account for numerical errors providing non unitary roots, we can take the argument of $(\zeta_n / |\zeta_n|)$). Then, supposing $\widetilde{\omega}_1 < \widetilde{\omega}_2 < \widetilde{\omega}_3 < \widetilde{\omega}_4$ (otherwise one can just order them), we have

$$\omega_1 = \frac{1}{2} (\widetilde{\omega}_4 + \widetilde{\omega}_2) \quad \omega_2 = \frac{1}{2} (\widetilde{\omega}_4 + \widetilde{\omega}_3)$$

6) In order to estimate α we rewrite (1) as

$$4 * y[n] = 4 * s[n] = \left(e^{i((\omega_1+\omega_2)n)} + e^{i((\omega_1-\omega_2)n)}\right) \alpha e^{i\Theta} + \left(e^{-i((\omega_1+\omega_2)n)} + e^{-i((\omega_1-\omega_2)n)}\right) \alpha e^{-i\Theta}.$$

So, having estimated ω_1 and ω_2, the solution of the linear system

$$\begin{bmatrix} \left(e^{i(\omega_1+\omega_2)} + e^{i(\omega_1-\omega_2)}\right) & \left(e^{-i(\omega_1+\omega_2)} + e^{-i(\omega_1-\omega_2)}\right) \\ \left(e^{i(\omega_1+\omega_2)2} + e^{i(\omega_1-\omega_2)2}\right) & \left(e^{-i(\omega_1+\omega_2)2} + e^{-i(\omega_1-\omega_2)2}\right) \end{bmatrix} \begin{bmatrix} \alpha e^{i\Theta} \\ \alpha e^{-i\Theta} \end{bmatrix} = \begin{bmatrix} y[1] \\ y[2] \end{bmatrix}$$

provides $\alpha e^{i\Theta}$ and $\alpha e^{-i\Theta}$, from which we can directly compute $|\alpha| = \alpha$ (having assumed the latter to be positive). The latter assumption can be relaxed and the sign of alpha computed using $\alpha e^{i\Theta}$ and $\alpha e^{-i\Theta}$.

5) The spectrum depicted in the picture presents a smooth curve. In addition, the goal of the spectrum estimation is to compute the surface under the spectral curve over determined intervals (and not to identify punctual frequencies). Therefore the optimal parametric approach is the Yule Walker one (based on the Yule Walker equations).

6) – Fix the order of the AR. The typical spectrum depicted in the figure suggest to take at least and AR of order 2. Therefore

$$RR[n] = p_1 RR[n-1] + p_2 RR[n-2] + W[n]$$

with $P(z) = 1 - p_1 z^{-1} - p_2 z^{-2}$ (and $RR[n]P(z) = W[n]$).

– Given the samples $rr[1], \dots, rr[10000]$, estimate the correlation using an empirical statistics

$$\widetilde{R}[k] = \frac{1}{10000-k} \sum_{n=1}^{10000-k} rr[n+k] rr[n]\,, \quad k = 0, \dots, (10000-1)\,,$$
$$\widetilde{R}[k] = \widetilde{R}[-k]\,, \quad k = -(10000-1), \dots, -1\,.$$

– Setup the Yule-Walker equations

$$\begin{bmatrix} \widetilde{R}[0] & \widetilde{R}[1] \\ \widetilde{R}[1] & \widetilde{R}[0] \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \widetilde{R}[1] \\ \widetilde{R}[2] \end{bmatrix}$$

– Solve the linear system to obtain p_1 and p_2, and therefore $P(z) = 1 - p_1 z^{-1} - p_2 z^{-2}$.

– Compute the noise variance using $\sigma^2 = \widetilde{R}[0] - p_1 \widetilde{R}[1] - p_2 \widetilde{R}[2]$.

– Obtain the estimate of the spectrum as

$$S(\nu) = \frac{\sigma^2}{P(e^{i2\pi\nu})} = \frac{\sigma^2}{1 - p_1 e^{-i2\pi\nu} - p_2 e^{-i4\pi\nu}}.$$

Exercise 4. LED LIGHT BULB ANALYSIS (18 PTS)

Your work for a company producing LED light bulbs. Being extremely concerned by providing a high quality product, your boss ask you to perform an analysis so to check if the LED light bulbs produced over a day have the same properties.

Every day the company produces 10000 light bulbs. To analyze the properties you flash (turn on and then off) **each** LED light bulb for one second and you record the emitted light sampled at 100Hz, obtaining $y[1], \dots, y[100]$. For each recording you compute:

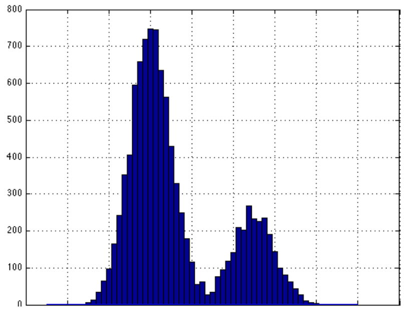
- the maximum emitted light intensity
- the maximum spectral energy
- the frequency of the maximum spectral energy
- the mean of the emitted light intensity

that is 6 characteristics that we shall call $c[1], \ldots, c[4]$

In order to reduce the complexity of the analysis and to verify whether it is really necessary to consider 4 characteristic, we use principal component analysis.

1) Describe in detail the principal component analysis method, step by step, from the **characteristics computed from the recorded emitted light** to the principal components, like if each step has to be interpreted and executed by a computer (in particular the input, the executed operation with corresponding equations, and the output of each step has to be clear). Also, **clearly indicate the dimensions of the matrices and vectors**.

After analyzing the variance of the principal components, it clearly appears that 1 principal component, namely $z_m[1]$, accounts for most of the total sample variation. When making an histogram of all the 10000 values of the principal component, the plots appears as a mixture of two distributions, as depicted in the figure below.



This suggests that the daily produced 10000 LED bulbs can be divided into two classes, each being described by a Gaussian distribution.

2) Provide the **explicit** expression of a two Gaussian mixture model for the principal components having as realization $z_m[1]$, $m = 1, \ldots, 10000$.

3) Write the corresponding likelihood function.

4) Why we cannot find an explicit expression for the parameter value maximizing the likelihood function?

5) How can we find the two values of the principal component each characterizing one of the two classes of the daily produced LED light bulbs?

Solution 4.

1) The dimension of the data is the number of identified shapes, that is M=10000, while the variables are the N=4 characteristics (maximum emitted light intensity, maximum spectral energy, frequency of the maximum spectral energy, mean of the emitted light intensity).

In order to be able to apply the PCA, the variable vector must be w.s.s. and centred.

We call $\boldsymbol{c}_m = [c_m[1], \dots, c_m[4]]^T$ the variables of the problem, where $m = 1, \dots, 10000$ indicates the analyzed bulb.

- Center the variable vector $c_m[1], \dots, c_m[4]$, for every $m = 1, \dots, 10000$,

$$c_m[k] = c_m[k] - \frac{1}{10000} \sum_{l=1}^{10000} c_l[k] \,, \quad k = 1, \dots, 4 \quad m = 1, \dots, 10000 \,.$$

- Compute the empirical covariance Matrix

$$\widehat{\boldsymbol{R}}_{\boldsymbol{c}} = \frac{1}{10000} \sum_{m=1}^{10000} \boldsymbol{c}_m * \boldsymbol{c}_m^H = \frac{1}{10000} \boldsymbol{C} * \boldsymbol{C}^H, \, (4 \times 4) \,,$$

  Where, $\boldsymbol{C} = \begin{bmatrix} \boldsymbol{c}_1 & \dots & \boldsymbol{c}_{10000} \end{bmatrix}, (4 \times 10000)$.

- Compute the unitary matrix of eigenvectors $\boldsymbol{V}$ (4×4) of $\widehat{\boldsymbol{R}}_{\boldsymbol{c}}$ and the corresponding eigenvalues $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \dots, \lambda_4)$ (4×4). That is, the solution of the equation

$$\widehat{\boldsymbol{R}}_{\boldsymbol{c}} \boldsymbol{V} = \boldsymbol{V} \boldsymbol{\Lambda} \,.$$

  We assume here the eigenvalues to be ordered from the highest to the lowest value (the eigenvectors are also ordered according to the corresponding eigenvalue).

- Identify the eigenvalues that clearly accounts for most of the total sample variation (energy), say $\lambda_1, \dots, \lambda_k$.

- Compute the matrix of the principal components

$$\boldsymbol{Z} = \boldsymbol{V}^T \boldsymbol{C} \,, (4 \times 10000) \,, \quad \text{with with N=4 rows and M=10000 columns} \,.$$

  The eigenvalues represents the variance of the principal components.

- Extract from $\boldsymbol{Z}$ the firs k rows which correspond to the k eigenvalues accounting for most of the total sample variation. The so obtained k principal components $z_m[1], \dots, z_m[k]$, $m = 1, \dots, 10000$, represents the essential characteristics of the data.

  In our case (see the following of the problem), k=1.

2) We assume $z_m[1]$ to be i.i.d., w.r.t. the index m (notice that this is not straigthtforward) and we model it as an i.i.d. two Gaussian mixture model. The model reads

$$f_Z(z_1[1], \ldots, z_{10000}[1]) = \prod_{m=1}^{10000} \left(\pi_1 \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(z_m[1]-\mu_1)^2}{2\sigma_1^2}} + \pi_2 \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(z_m[1]-\mu_2)^2}{2\sigma_2^2}} \right)$$

where

- π_1 and π_2 are the probability the $z_m[1]$ comes from the first and the second class, respectively.
- μ_1 and μ_2 represents the mean of the first and the second class, respectively. These corresponds to the center of the two bells of the histogram.
- σ_1 and σ_2 represents the standard deviation of the first and the second class, respectively. These corresponds to the spread of the two bells of the histogram.

3) Given a realization of the process $z_1[1], \ldots, z_{10000}[1]$, the likelihood function is given by the probability density function, but considered as a function of the parameters of the model

$$h(\pi_1, \pi_2, \sigma_1^2, \sigma_2^2, \mu_1, \mu_2 \, ; z_1[1], \ldots, z_{10000}[1]) = \prod_{m=1}^{10000} \left(\pi_1 \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(z_m[1]-\mu_1)^2}{2\sigma_1^2}} + \pi_2 \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(z_m[1]-\mu_2)^2}{2\sigma_2^2}} \right)$$

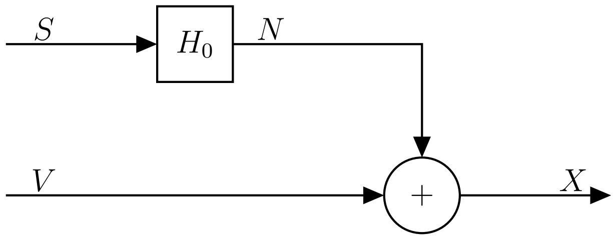
4) The likelihood is a products of sums. Even if we take the logarithm we end up with sum of logarithms of a sum. Therefore it is not possible to find an explicit solution to the parameter estimation problem.

5) The values of the two principal components are the center of the two bells of the histograms, that is, the means of the two Gaussian distributions. These are estimate as argmax of the likelihood function (values maximizing the likelihood function). They can be obtained (together with the other parameters) using the EM algorithm.

Exercise 5. JANE, JIM AND PRINCE CHARMING (20 PTS)

Jane is a Master student in Communication Systems at EPFL. Her boyfriend, Prince Charming, left for an internship in a far-far-away country. Because they cannot live without seeing each-other, they decided to organize video-conferences every evening. This seems to work fine, but there is still one problem disturbing these meetings; and the name of this problem is Jim, Jane's younger brother. Every time Jane and Prince Charming are talking, he starts screaming in his room.

This scenario is depicted in the figure below: the microphone captures both Jane's voice, V, and Jim's screams, S, filtered by the acoustic channel, $H_0 = a_0 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}$ (where H_0 is a stable filter) . The length of the filter is known, but the coefficients a_i, are unknown.



The screams are annoying for Prince Charming, but Jane has the solution: last year she took the Statistical Signal Processing Class, and her favorite part was the noise-canceling. So, she decided to implement a system that will remove Jim's screams from the conversation.

Firstly, let us assume that Jane's voice, V, and Jim's screams, S, are wide sense stationary and independent. Also, S can be modeled by a centered Gaussian white noise with variance σ_S^2.

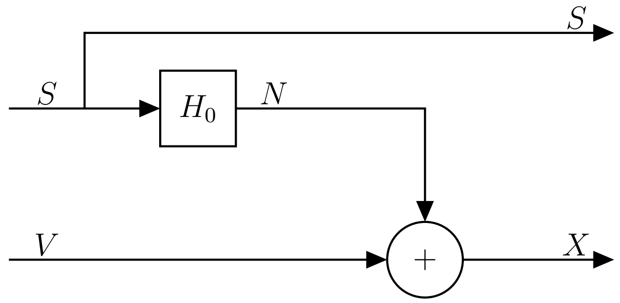
1) Compute the autocorrelation of N.

2) Describe a method that will remove in a MSE sense the noise N from X.

3) After implementing the previous method, Jane realizes that S and V are stationary only for durations of 100 ms. How would you implement an optimal filter in this case?

4) The implementation of the previous question is too slow for real-time processing. Explain why this is the case.

5) Assume now that Jane does not know anything about the statistics of S, but she is able to place a microphone in her brother's room. Describe how to use an adaptive filter to remove the noise N from the microphone signal, X. Draw where you would place the filter, what should the filter length be, and what signals you would use to determine the filter coefficients.



6) Now assume that the LMS algorithm is to be used for computing the adaptive filter.

Write the expression for updating the filter coefficients. What is the possible range for the step-size, so that the algorithm converges?

Solution 5.

1) We have

$$N[n] = \sum_{i=0}^{N} a_i S[n-i] \, .$$

S is wide sense stationary and H_0 is stable, then, by the fundamental filtering formula, N is also wide sense stationary. Its correlation reads

$$\begin{aligned} R_N[k] &= E[N[n]N^*[n-k]] \\ &= E\left[\sum_{i=0}^{N} a_i S[n-i] \sum_{j=0}^{N} a_j^* S^*[n-j-k]\right] \\ &= \sum_{i=0}^{N}\sum_{j=0}^{N} a_i a_j^* E\left[S[n-i]S^*[n-j-k]\right] \\ &= \sum_{i=0}^{N}\sum_{j=0}^{N} a_i a_j^* \delta_{i-j-k}\sigma_S^2 \\ &= \sum_{i=k}^{N} a_i a_{i-k}^* \sigma_S^2 \end{aligned}$$

The last equality follows from the fact that δ_{i-j-k} is non-zero for $j = i-k$, and $j \geq 0$, therefore $i \geq k$.

2) - The signals are wide sense stationary, therefore, we are in the framework of Wiener filtering. The available signal is X and the signal we want to estimate is V, where $X[n] = V[n] + N[n]$.

- The Wiener filter reads

$$H_1(\omega) = \frac{S_{VX}(\omega)}{S_X(\omega)}\,.$$

where, given the independence of V and S, $S_{VX}(\omega) = S_V(\omega)$. Notice that we do not have access to V and therefore to S_V. Consequently, using $X[n] = V[n] + N[n]$, we express $S_V(\omega) = S_X(\omega) - S_N(\omega)$. Finally

$$H_1(\omega) = \frac{S_X(\omega) - S_N(\omega)}{S_X(\omega)}\,.$$

- $S_X(\omega)$ is obtained directly form the samples of $X[n]$ using a periodogram or another appropriate spectral estimator, while $S_N(\omega)$ is estimate as the power of the uncorrelated component of X (whitening filter) or from $R_N[k]$, determined before.

Alternatively, we can determine the filter as

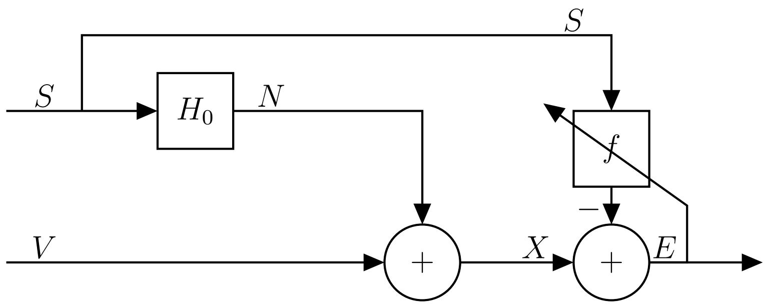
$$\mathbf{H_1} = \mathbf{R_X}^{-1}\mathbf{R_{VX}} = \mathbf{R_X}^{-1}(\mathbf{R_X} - \mathbf{R_N}).$$

$R_X[k]$ can be obtained from the samples of X, and $R_N[k]$ was determined before.

3) The filter coefficients should be computed every 100 ms. One must be careful that now the maximum length of the filter is $0.1 \times Fs$, where Fs is the sampling frequency.

4) Estimating the power spectrum $S_X(\omega)$ (or computing the inverse of R_X) is too computationally demanding.

5) The filter order is N (the filter length $N + 1$), and the error signal controlling the filter is $E[n] = X[n] - (S * f)[n]$, where $*$ represents convolution.



6)

$$\mathbf{f}_{n+1} = \mathbf{f}_n + \mu\mathbf{S}_n E[n]$$

$\mathbf{f}_{n+1}$ and $\mathbf{f}_n$ are $(N+1)\times 1$ vectors containing the filter coefficients at time instants $n+1$ and n, μ is the step size, $E[n]$ is the error signal (to be minimized) and $\mathbf{S}$ is a $(N+1)\times 1$ vector containing the last $N+1$ samples of S.

To insure convergence, the step size should be $0 < \mu < \frac{2}{\lambda_{max}}$, where λ_{max} is the highest eigenvalue of R_S, and R_S can be estimated from the samples of S.

An easier-to-determine condition is $0 < \mu < \frac{2}{(N+1)\sigma_S^2}$, because it is easier to estimate σ_S^2 than the eigenvalues of R_S (where $N + 1$ is the length of the filter).

Grade Scale.

The exams accounts for a total of 75 points (exact response to each question).

The grading has been done on a 60 points scale (60 points = 6/6), according to the following formula

$$\text{grade over 6} = 1 + (5 * \text{points}/60)$$

and then rounded to .5 steps, that is

$$\text{rounded grade over 6} = (\text{round-to-0-digit}(2 * \text{grade over 6}))/2$$

The result is then constraint to be at maximum 6.