

Stochastic ProcessesW.S.S.

$$E[X[n]] = \text{const.}, \text{Var}(X[n]) = \text{const.} < \infty$$

$$E[X[k]X^*[l]] = R(k-l), \forall k, l \in \mathbb{Z},$$

PSD: w.s.s. + $R_X(k) \in \ell_1$ (summable)

$$S_X(\omega) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j\omega k}.$$

Fundamental Filtering Formula

$X[n]$ w.s.s., $R_X(k) \in \ell_1$, and $h_k \in \ell_1$, then

$$Y[n] = \sum_{k=-\infty}^{\infty} h_{n-k} X[k], \text{ is w.s.s. with}$$

$$E[Y] = E[X] \sum_{k=-\infty}^{\infty} h_k, \quad R_Y(k) \in \ell_1$$

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega), H(\omega) = \text{DTFT of } h_k$$

Markov Chain

$\{X[n]\}_{n \in \mathbb{Z}}$ (considered stationary) taking discrete values in $\mathcal{D}$, |

$$P(X[n] = i_n \mid X[n-1] = i_{n-1}, X[n-2] = i_{n-2}, \dots)$$

$$= P(X[n] = i_n \mid X[n-1] = i_{n-1}), \forall i_n, i_{n-1}, \dots \in \mathcal{D}$$

Bayes' Rule

A and B with discrete values in $\mathcal{D}$,

$$P(A=k \mid B=l) = \frac{P(A=k, B=l)}{P(B=l)}, \quad k, l \in \mathcal{D}.$$

AR Process

A w.s.s. process $X[n]$, with values in $\mathbb{R}$, |

$$\sum_{k=0}^M p_k X[n-k] = W[n], \quad n \in \mathbb{Z},$$

$W[n]$, is a zero mean Gaussian white noise

$p_k, k=0, \dots, M$ bounded coefficients (real or complex). We assume $p_0=1$.

Filtering Interpretation (z^{-1} delay operator)

$$P(z)X[n] = W[n]$$

Canonical form: $P(z)$ strict. min. phase, $p_0=1$.

Correlation:

$$R_X[m] + \sum_{k=1}^{M-1} p_k R_X[m-k] = \delta_m \sigma_W^2, \quad m \geq 0.$$

PSD (fundamental filtering formula):

$$S_X(\omega) |P(e^{j\omega})|^2 = \sigma_W^2.$$

Harmonic Processes

$$X[n] = \sum_{k=1}^K \alpha_k e^{j(\omega_k n + \Theta_k)}, n \in \mathbb{N},$$

Θ_k i.i.d. uniformly distributed over $[0, 2\pi]$.

$$R_X[l] = \sum_{k=1}^K |\alpha_k|^2 e^{j\omega_k l}, \quad S_X(\omega) = \sum_{k=1}^K |\alpha_k|^2 \delta(\omega - \omega_k).$$

Hilbert SpacesProjection Theorem

E, S Hilbert spaces with $S \subseteq E$, then

$$\forall v \in E, \exists! b \in S$$

$$b = \arg \min_{c \in S} \|v - c\|, \langle v - b, c \rangle = 0, \forall c \in S,$$

Projection Theorem w.s.s.

E, S Hilbert spaces of w.s.s. processes with $S \subseteq E \subseteq L^2(P)$, then

$$\forall X[n] \in E, \exists! Y[n] \in S$$

$$Y[n] = \arg \min_{U[n] \in S} E[|X[n] - U[n]|^2],$$

$$E[(X[n] - Y[n])U^*[n]] = 0, \forall U[n] \in S,$$

Empirical StatisticsBias & Variance

$\hat{S}(x[1], \dots, x[N])$ empirical statistics of a probabilistic moment s .

Bias $E[\hat{S}(X[1], \dots, X[N])] - s$,

Variance $\text{Var}(\hat{S}(X[1], \dots, X[N]) - s)$.

Unbiased & Biased Correlation

$$\hat{R}_X^{\text{NB}}(k) = \frac{1}{N-|k|} \sum_{n=1}^{N-|k|} x[n+k]x^*[n],$$

$$\hat{R}_X^{\text{B}}(k) = \frac{1}{N} \sum_{n=1}^{N-|k|} x[n+k]x^*[n].$$

MethodsLinear Estimation of w.s.s.: Wiener Filter

Estimation of $X[n]$ given $Y[n]$

Normal equations $R_{XY}[u] = \sum_{m \in \mathbb{Z}} h[m] R_Y[u-m]$

Wiener Filter $H(e^{j\omega}) = \frac{S_{XY}(\omega)}{S_Y(\omega)}$

Linear Prediction of w.s.s.: Yule-Walker

Prediction of $X[n]$ as linear combination of $X[n-1], \dots, X[n-N]$.

Coefficients a_k solutions of

$$\begin{bmatrix} R_X[0] & \dots & R_X[N-1] \\ \vdots & \ddots & \vdots \\ R_X[N-1] & \dots & R_X[0] \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} R_X[1] \\ \vdots \\ R_X[N] \end{bmatrix}.$$

Linear Estimation of AR: Yule-Walker

$\sum_{k=0}^N p_k X[n-k] = W[n]$. Coeff. p_k solution of

$$-\begin{bmatrix} R_X[0] & \dots & R_X[N-1] \\ \vdots & \ddots & \vdots \\ R_X[N-1] & \dots & R_X[0] \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_N \end{bmatrix} = \begin{bmatrix} R_X[1] \\ \vdots \\ R_X[N] \end{bmatrix}.$$

$$\sigma_W^2 = R_X[0] + R_X[1]p_1 + \dots + R_X[N]p_N$$

Linear Prediction of AR: Projection Theorem

$$H(X, n) = H(W, n), \quad \forall n \in \mathbb{Z}.$$

Intuitive property:

$$Y \in H(X, n+k), Y = A+B, A \perp H(X, n), B \in H(X, n)$$

orthogonal projection of Y onto $H(X, n)$ is B .

Estimation Param. Prob.: MLE

θ parameters of the prob. function $f_Y, y[1], \dots, y[N]$ realization of the process $Y[n]$, then

$$\hat{\theta} = \arg \max_{\theta} f_Y(y[1], \dots, y[N], \theta)$$

Spectral EstimationPeriodogram: General w.s.s. process

$$P_X^N(\omega) = \frac{1}{N} \left| \sum_{n=1}^N x[n] e^{-j\omega n} \right|^2 = \frac{1}{N} |\hat{x}_N(\omega)|^2,$$

$$\text{Bias } \sum_{k=-N+1}^{N-1} \frac{N-|k|}{N} R_X[k] e^{-j\omega k} - S_X(\omega)$$

Variance constant

Resolution $\Delta f > \frac{1}{N}$

Annihilating Filter: Line Spectra

Estimation of line spectrum frequencies and amplitudes of a Harmonic w.s.s. process in absence of noise

1) Given $2K$ observations, solve the system

$$\begin{bmatrix} x[K-1] & \dots & x[0] \\ \vdots & & \vdots \\ x[2K-2] & \dots & x[K-1] \end{bmatrix} \begin{bmatrix} h[1] \\ \vdots \\ h[K] \end{bmatrix} = - \begin{bmatrix} x[K] \\ \vdots \\ x[2K-1] \end{bmatrix}$$

2) Compute $H(z)$ and the zeros of $H(z)$

3) Compute the argument of the zeros of $H(z)$

4) Compute ω_k from the zeros' arguments

5) Compute the amplitudes $|\alpha_k|^2$ by solving

$$\begin{bmatrix} 1 & \dots & 1 \\ e^{j\omega_1} & \dots & e^{j\omega_K} \\ \vdots & & \vdots \\ e^{j\omega_1(K-1)} & \dots & e^{j\omega_K(K-1)} \end{bmatrix} \begin{bmatrix} \alpha_1 e^{j\Theta_1} \\ \alpha_2 e^{j\Theta_2} \\ \vdots \\ \alpha_K e^{j\Theta_K} \end{bmatrix} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[K-1] \end{bmatrix}$$

MUSIC: Line Spectra

Estimation of line spectrum frequencies and amplitudes of a Harmonic w.s.s. process in the presence of noise

1) Given M observations with $M \gg N > K$ center the process and compute the empirical correlation matrix

$$\hat{\mathbf{R}}_{\mathbf{Y}}^{NN} = \frac{1}{M-N+1} \sum_{n=1}^{M-N+1} \mathbf{y}^{N1}[n] \mathbf{y}^{N1}[n]^H;$$

2) Compute the eigendecomposition $\hat{\mathbf{G}}^{N(N-K)}$ of $\hat{\mathbf{R}}_{\mathbf{Y}}^{NN}$.

3.a) Determine the peaks of

$$\frac{1}{e^{N1(\omega)^H} \hat{\mathbf{G}}^{N(N-K)} \hat{\mathbf{G}}^{N(N-K)} e^{N1(\omega)}};$$

3.b) Determine the minimum values of

$$e^{N1(\omega)^H} \hat{\mathbf{G}}^{N(N-K)} \hat{\mathbf{G}}^{N(N-K)} e^{N1(\omega)}.$$

4) Compute the modulus of the amplitudes using $\mathbf{R}_{\mathbf{Y}}^{NN} = \mathbf{E}^{NK} \mathbf{A}^{KK} \mathbf{E}^{NKH} + \sigma_W^2 \mathbf{I}^{NN}$.

MUSIC: Smooth Spectra

1) Given N observations with $N \gg M$ center the process and compute the empirical correlation

$$\hat{R}_X[k] = \frac{1}{N} \sum_{n=1}^{N-k} x_0[n+k] x_0[n]^*, \quad k=0, \dots, M$$

2) Solve the Yule Walker equations to obtain $\hat{p}_1, \dots, \hat{p}_M$

3) Compute the estimate of the spectrum as

$$\hat{S}_X(\omega) = \frac{\hat{\sigma}_W^2}{|P(z)|^2} \Big|_{z=e^{j\omega}}.$$