PROBLEM 2. 20 points (Paper and Pencil + MATLAB/Python)

1. By the definition of the IDFT, we have

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{N}kn}, \quad n = 0, \cdots, N-1.$$

But the symbols $X_k = 1 + j$, for $k = 0, \cdots, N-1$, so the previous formula becomes:

$$x_n = \frac{1}{N}(1+j)\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kn}, \quad n = 0, \cdots, N-1.$$

For $n = 0$, we obtain $x_0 = 1 + j$. Otherwise, for $n = 1, \cdots, N-1$ we have

$$x_n = \frac{1}{N}(1+j)\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kn} = \frac{1}{N}(1+j)\frac{e^{j\frac{2\pi}{N}Nn} - 1}{e^{j\frac{2\pi}{N}n} - 1} = 0.$$

So the last $L = 20$ values of the training sequence OFDM block are all 0, which means that the CP of this OFDM block contains only zero values.

2.
$$y_n = \sum_{i=0}^{P-1} h_i x_{n-i} + z_n, \quad n = 0, \ldots, Q+P-2.$$

3.
$$\boldsymbol{y} = C\boldsymbol{h} + \boldsymbol{z},$$

where C is an $Q + P - 1 \times P$ matrix:

$$C = \begin{pmatrix} x_0 & 0 & \ldots & 0 \\ x_1 & x_0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ x_{Q-1} & x_{Q-2} & \ldots & x_{Q-P} \\ 0 & x_{Q-1} & \ldots & x_{Q-P+1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & x_{Q-1} \end{pmatrix}.$$

4. The values of the CP of the training sequence OFDM block are affecting the entries of the last $P - 1$ rows of C (the lower-left corner consisting of all-zeros). So if they were not 0, we should have taken them into account when writing down the matrix C.

5. We are seeking for the $\hat{\boldsymbol{h}}$ which minimizes

$$\|\boldsymbol{y} - C\boldsymbol{h}\|^2$$

over all $\boldsymbol{h}$. $C\hat{\boldsymbol{h}}$ is an element of the inner-product space spanned by the columns of C. So $\mathcal{V} = \text{span}\{C_1, \ldots, C_P\}$ where C_i is the ith column of C, $i = 1, 2, \ldots, P$. We are seeking the vector $\hat{\boldsymbol{y}} \in \mathcal{V}$ which minimizes

$$\|\boldsymbol{y} - \hat{\boldsymbol{y}}\|^2.$$

The projection theorem tells us that $\hat{\boldsymbol{y}}$ is the projection of $\boldsymbol{y}$ into $\mathcal{V}$. It has the property that the error vector $\boldsymbol{y} - \hat{\boldsymbol{y}}$ is orthogonal to every element of $\mathcal{V}$. In particular, it is orthogonal to the columns of C. Hence

$$\langle \boldsymbol{y} - C\hat{\boldsymbol{h}}, C_i \rangle = 0, \quad i = 1, 2, \ldots, P.$$

Hence

$$\langle \boldsymbol{y}, C_i \rangle = \langle C\hat{\boldsymbol{h}}, C_i \rangle, \quad i = 1, 2, \ldots, P.$$

Equivalently,

$$C_i^\dagger \boldsymbol{y} = C_i^\dagger C\hat{\boldsymbol{h}} \quad i = 1, 2, \ldots, P.$$

In matrix form

$$C^\dagger \boldsymbol{y} = C^\dagger C\hat{\boldsymbol{h}}.$$

Solving for $\hat{\boldsymbol{h}}$ yields the least-squares approximation of $\boldsymbol{h}$:

$$\hat{\boldsymbol{h}} = \left(C^\dagger C\right)^{-1} C^\dagger \boldsymbol{y}.$$

6. See the provided routines. One can observe a very good agreement between $\hat{\boldsymbol{h}}$ and $\boldsymbol{h}$.