ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 22 Modern Digital Communications
Final Solutions December 21, 2022

PROBLEM 1. 18 points (Paper and Pencil)

1. By the linearity of convolution, we have

$$r(t) = s(t) \star h(t) + N(t) = \sum_k s[k](p \star h)(t - kT) + N(t) = \sum_k s[k]g(t - kT) + N(t),$$

   where $g(t) = (p \star h)(t)$.

2. Again, by the linearity of the convolution, we have

$$y(t) = r(t) \star q(t) = \sum_k s[k](p \star h \star q)(t - kT) + \int N(\alpha)q(t - \alpha)d\alpha.$$

   Hence,

$$y(t) = \sum_k s[k]f(t - kT) + \int N(\alpha)q(t - \alpha)d\alpha,$$

   where $f(t) = (p \star h \star q)(t)$.

3. $$y[k] = y(kT) = \sum_j s[j]f(kT - jT) + \int N(\alpha)q(kT - \alpha)d\alpha.$$

   Hence,

$$y[k] = \sum_n \underbrace{s[k-n]}_{=:b_n} \underbrace{(p \star h \star q)(nT)}_{=:v[n]} + z[k],$$

   where $z[k] = \int N(\alpha)q(kT - \alpha)d\alpha$.

4. Since $N(t)$ is a zero-mean white Gaussian process, $z[k]$ is a collection of zero-mean jointly Gaussian random variables with covariance

$$E\big[z[k]z[j]^*\big] = N_0\langle q(kT - t), q(jT - t)\rangle.$$

5. Under the new assumptions,

$$v[n] = (p \star q)(nT) = \int p(\alpha)q(nT - \alpha)d\alpha = \int p(\alpha)p^*(\alpha - nT)d\alpha = \langle p(\alpha), p(\alpha - nT)\rangle = \delta_n.$$

   Hence, $y[k] = s[k] + z[k]$, that is, there is no ISI (Inter-Symbol Interference).

$$E\big[z[k]z[j]^*\big] = N_0\langle q(kT - t), q(jT - t)\rangle = N_0\langle p(t - jT), p(t - kT)\rangle = N_0\delta_{j-k}.$$

   Hence, the noise process will have independent components.

6. We have seen in class that $p_{\mathcal{F}}(f)$ is a rectangle of width $\frac{1}{T}$ with height $\sqrt{T}$, so it satisfies the Nyquist criterion, that is:

$$\text{l.i.m.} \sum_{k=-\infty}^{\infty} \left| p_{\mathcal{F}}(f - \frac{k}{T}) \right|^2 = T \quad \text{for } f \in \mathbb{R}.$$

Problem 3. 18 points (Paper and Pencil + MATLAB/Python)

1. We have shown in class that the LS estimate of $\boldsymbol{a}$ is $\hat{\boldsymbol{a}}_{\texttt{LS}} = \left(D^\dagger D\right)^{-1} D^\dagger \boldsymbol{y}$. Since D is a non-singular diagonal matrix, $\left(D^\dagger D\right)^{-1} = D^{-1}\left(D^\dagger\right)^{-1}$, so the previous expression simplifies to
$$\hat{\boldsymbol{a}}_{\texttt{LS}} = D^{-1}\boldsymbol{y}.$$

2. We have shown in class that the LMMSE estimate of $\boldsymbol{a}$ is $\hat{\boldsymbol{a}}_{\texttt{LMMSE}} = K_{\boldsymbol{ay}} K_{\boldsymbol{y}}^{-1}\boldsymbol{y}$. We can easily compute the covariance matrices:
$$K_{\boldsymbol{ay}} = E\left[\boldsymbol{a}\boldsymbol{y}^\dagger\right] = E\left[\boldsymbol{a}(D\boldsymbol{a}+\boldsymbol{z})^\dagger\right] = E\left[\boldsymbol{a}\boldsymbol{a}^\dagger\right] D^\dagger = D^\dagger \ ,$$
and
$$K_{\boldsymbol{y}} = E\left[\boldsymbol{y}\boldsymbol{y}^\dagger\right] = E\left[(D\boldsymbol{a}+\boldsymbol{z})(D\boldsymbol{a}+\boldsymbol{z})^\dagger\right] = DE\left[\boldsymbol{a}\boldsymbol{a}^\dagger\right] D^\dagger + E\left[\boldsymbol{z}\boldsymbol{z}^\dagger\right] = DD^\dagger + \sigma^2 I_N \ .$$
Hence,
$$\hat{\boldsymbol{a}}_{\texttt{LMMSE}} = D^\dagger(DD^\dagger + \sigma^2 I_N)^{-1}\boldsymbol{y}.$$