ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 22 — Modern Digital Communications

Final Solutions — December 22, 2021

PROBLEM 1. 20 points (Paper and Pencil)

1. S is an $N \times t$ matrix:
$$S = \begin{pmatrix} x_1[1] & x_2[1] & \dots & x_t[1] \\ x_1[2] & x_2[2] & \dots & x_t[2] \\ \vdots & \vdots & \ddots & \vdots \\ x_1[N] & x_2[N] & \dots & x_t[N] \end{pmatrix}.$$

2. We are seeking for the $\hat{\boldsymbol{h}}_p$ which minimizes
$$\|\boldsymbol{y}_p - S\boldsymbol{h}_p\|^2$$
over all $\boldsymbol{h}_p$.

3. $S\hat{\boldsymbol{h}}_p$ is an element of the inner-product space spanned by the columns of S. So $\mathcal{V} = \text{span}\{S_1, \dots, S_t\}$ where S_i is the ith column of S, $i = 1, 2, \dots, t$.

4. We are seeking the vector $\hat{\boldsymbol{y}}_p \in \mathcal{V}$ which minimizes
$$\|\boldsymbol{y}_p - \hat{\boldsymbol{y}}_p\|^2.$$
The projection theorem tells us that $\hat{\boldsymbol{y}}_p$ is the projection of $\boldsymbol{y}_p$ into $\mathcal{V}$. It has the property that the error vector $\boldsymbol{y}_p - \hat{\boldsymbol{y}}_p$ is orthogonal to every element of $\mathcal{V}$. In particular, it is orthogonal to the columns of S. Hence
$$\langle \boldsymbol{y}_p - S\hat{\boldsymbol{h}}_p, S_i \rangle = 0, \quad i = 1, 2, \dots, t.$$
Hence
$$\langle \boldsymbol{y}_p, S_i \rangle = \langle S\hat{\boldsymbol{h}}_p, S_i \rangle, \quad i = 1, 2, \dots, t.$$
Equivalently,
$$S_i^\dagger \boldsymbol{y}_p = S_i^\dagger S\hat{\boldsymbol{h}}_p \quad i = 1, 2, \dots, t.$$
In matrix form
$$S^\dagger \boldsymbol{y}_p = S^\dagger S\hat{\boldsymbol{h}}_p.$$

5. Solving for $\hat{\boldsymbol{h}}_p$ yields the least-squares approximation of $\boldsymbol{h}_p$:
$$\hat{\boldsymbol{h}}_p = \left(S^\dagger S\right)^{-1} S^\dagger \boldsymbol{y}_p.$$

6. We have shown in class that the LMMSE estimate of $\boldsymbol{h}_p$ is $\hat{\boldsymbol{h}}_p = K_{\boldsymbol{h}_p\boldsymbol{y}_p} K_{\boldsymbol{y}_p}^{-1} \boldsymbol{y}_p$. We can easily compute the covariance matrices:
$$K_{\boldsymbol{h}_p\boldsymbol{y}_p} = E\left[\boldsymbol{h}_p \boldsymbol{y}_p^\dagger\right] = E\left[\boldsymbol{h}_p (S\boldsymbol{h}_p + \boldsymbol{z}_p)^\dagger\right] = E\left[\boldsymbol{h}_p \boldsymbol{h}_p^\dagger\right] S^\dagger = S^\dagger\ ,$$
and
$$K_{\boldsymbol{y}_p} = E\left[\boldsymbol{y}_p \boldsymbol{y}_p^\dagger\right] = E\left[(S\boldsymbol{h}_p + \boldsymbol{z}_p)(S\boldsymbol{h}_p + \boldsymbol{z}_p)^\dagger\right] = SE\left[\boldsymbol{h}_p \boldsymbol{h}_p^\dagger\right] S^\dagger + E\left[\boldsymbol{z}_p \boldsymbol{z}_p^\dagger\right] = SS^\dagger + \sigma^2 I_N.$$
Hence,
$$\hat{\boldsymbol{h}}_{p,\texttt{LMMSE}} = S^\dagger (SS^\dagger + \sigma^2 I_N)^{-1} \boldsymbol{y}_p.$$

7. If $\sigma^2 \approx 0$ (or $\sigma^2 \ll \min(\text{diag}(SS^\dagger))$, $\hat{\boldsymbol{h}}_{p,\texttt{LMMSE}} \approx S^\dagger (SS^\dagger)^{-1} \boldsymbol{y}_p$. Since $N = t$, S is a square matrix. Then, $(SS^\dagger)^{-1} = (S^\dagger)^{-1} S^{-1}$, and $\hat{\boldsymbol{h}}_{p,\texttt{LMMSE}} \approx S^{-1}\boldsymbol{y}_p$. On the other hand, when $N = t$, $\hat{\boldsymbol{h}}_p = \left(S^\dagger S\right)^{-1} S^\dagger \boldsymbol{y}_p = S^{-1} (S^\dagger)^{-1} S^\dagger \boldsymbol{y}_p = S^{-1}\boldsymbol{y}_p$. Hence, $\hat{\boldsymbol{h}}_{p,\texttt{LMMSE}} \approx \hat{\boldsymbol{h}}_p$.