(a)

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(b) Note: all functions of one binary variable will be represented as a vector of length 2 of the form $(f_i(0), f_i(1))$.

Initially, we know

$$\begin{aligned} F_{1\to 1} &= (\epsilon\,, 1-\epsilon) \\ F_{2\to 2} &= (1-\epsilon\,, \epsilon) \\ F_{3\to 3} &= (1-\epsilon\,, \epsilon) \end{aligned}$$

Next step:

$$\begin{aligned} V_{2\to 4} &= F_{2\to 2} = (1-\epsilon\,, \epsilon) \\ V_{3\to 5} &= F_{3\to 3} = (1-\epsilon\,, \epsilon) \end{aligned}$$

Next step:

$$\begin{aligned} F_{5\to 1}(x_1) &= \sum_{x_3} f_5(x_1, x_3) V_{3\to 5}(x_3) \\ &= f_5(x_1, x_1) V_{3\to 5}(x_1)) \\ &= V_{3\to 5}(x_1) \end{aligned}$$

where we used the fact that f_5 equals 1 if $x_3 = x_1$ and vanishes otherwise.

In terms of the vector notation we have

$$F_{5\to 1} = (1-\epsilon\,, \epsilon).$$

This says that the message that comes out from the factor node f_5 is the same as what goes in. The same is true for f_4.

Hence also

$$F_{4\to 1} = (1-\epsilon\,, \epsilon).$$

Next step:

$$\begin{aligned} V_{1\to 4} &= F_{5\to 1} F_{1\to 1} = (1-\epsilon\,, \epsilon)(\epsilon\,, 1-\epsilon) = (\epsilon(1-\epsilon)\,, \epsilon(1-\epsilon)) \\ V_{1\to 5} &= F_{4\to 1} F_{1\to 1} = (\epsilon(1-\epsilon)\,, \epsilon(1-\epsilon)) \end{aligned}$$

where we used the fact that in the vector notation when we multiply two functions we multiply component-wise the corresponding vectors.

Next step, proceeding similarly:

$$\begin{aligned} F_{4\to 2} &= (\epsilon(1-\epsilon)\,, \epsilon(1-\epsilon)) \\ F_{5\to 3} &= (\epsilon(1-\epsilon)\,, \epsilon(1-\epsilon)). \end{aligned}$$

To compute the "marginals" (in quotation since they are marginals up to a multiplicative constant) we multiply the messages into the corresponding variable node.

$$\sum_{\sim x_1} f = (\epsilon(1-\epsilon)^2\,, \epsilon^2(1-\epsilon))$$
$$\sum_{\sim x_2} f = (\epsilon(1-\epsilon)^2\,, \epsilon^2(1-\epsilon))$$
$$\sum_{\sim x_3} f = (\epsilon(1-\epsilon)^2\,, \epsilon^2(1-\epsilon))$$

(c) Since $\epsilon < 0.5$, $\epsilon(1-\epsilon)^2 > \epsilon^2(1-\epsilon)$ and $\hat{x}_1 = \hat{x}_2 = \hat{x}_2 = 0$

(d)

$$(f_1(0)\,, f_1(1)) = (1-\epsilon\,, \epsilon)$$
$$(f_2(0)\,, f_2(1)) = (\epsilon\,, 1-\epsilon)$$
$$(f_3(0)\,, f_3(1)) = (1-\epsilon\,, \epsilon)$$