Modern Digital Communications: A Hands-On Approach

Relationship Between Fourier Transform and DFT

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- Course material of Prof. Bixio Rimoldi -

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Orthonormal Expansions

Let us first review the Fourier transform and the related transforms. The place to start is the concept of **orthonormal expansion**.
A vector $\underline{x}$ (not necessarily an n-tuple) that belongs to an **inner-product space** spanned by an **orthonormal basis** $\underline{\phi}_1, \ldots, \underline{\phi}_N$ can be written as

$$\underline{x} = \sum_{i=1}^{N} a_i \underline{\phi}_i \qquad \text{where}$$

$$a_i = \langle \underline{x}, \underline{\phi}_i \rangle \overset{(n\text{-tuple})}{=} \sum_{k=1}^{N} x_k \phi_{ik}^*.$$

Here $\langle \cdot, \cdot \rangle$ denotes the **inner product** and * denotes **complex conjugation**.
The coefficient a_i may be interpreted as the **(signed) length** of the **projection** of $\underline{x}$ onto $\underline{\phi}_i$.

The transformation **preserves inner products** in the sense that

$$\langle \underline{x}, \underline{y} \rangle = \langle \underline{a}, \underline{b} \rangle,$$

for any two vectors $\underline{x}$ and $\underline{y}$ of the inner-product space and the corresponding n-tuples $\underline{a}$ and $\underline{b}$ of coefficients.

Hence it **preserves the norm**, i.e., $\|\underline{x}\| = \|\underline{a}\|$, where $\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle}$ and $\|\underline{a}\|$ is defined similarly.

Transformations that preserve norms are called **unitary**.

We refer to $\sum a_i \underline{\phi}_i$ as the **orthonormal expansion** of $\underline{x}$ (with respect to the basis $\underline{\phi}_1, \ldots, \underline{\phi}_N$).

The FT, FS, DFS, DFT but also the transformation between a band-limited signal and its samples are all "cousins" of the above orthonormal expansion (the basis is not always normalized though).

When the vectors are functions, the inner product is an integral (rather than a sum).

When the orthonormal basis consists of a continuum of functions, the expansion itself is an integral (rather than a sum).

Fourier Transform (FT)

$$s(t) = \int_{-\infty}^{\infty} s_{\mathcal{F}}(f) e^{j2\pi ft} df$$
$$s_{\mathcal{F}}(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} dt$$

Notice the symmetry of the formulas.

The "$-$" in the exponent is due to the complex conjugation of $e^{j2\pi ft}$.

Be aware that the time functions of the form $e^{j2\pi ft}$ do not have finite norm. Yet Parseval's relationship holds:

$$\int x(t) y^*(t) dt = \int x_{\mathcal{F}}(f) y_{\mathcal{F}}^*(f) df.$$

Fourier Series (FS)

Let $\tilde{s}(t)$ be periodic of period T_p.

$$\tilde{s}(t) = \sum_{k\in\mathbb{Z}} A_k e^{j\frac{2\pi}{T_p}kt}, \qquad t \in \mathbb{R}$$

$$A_k = \frac{1}{T_p}\int_0^{T_p} \tilde{s}(t) e^{-j\frac{2\pi}{T_p}kt} dt, \qquad k \in \mathbb{Z}.$$

Notice the simplicity of the expansion (top formula).
Be aware though that the FS basis is orthogonal but not orthonormal.
To make it orthonormal over one period we have to use

$$\phi_k(t) = \frac{1}{\sqrt{T_p}} \exp\left\{j\frac{2\pi}{T_p}kt\right\}$$

in the expansion and $\phi_k^*(t)$ in the projection.
The factor $1/T_p$ in the expression for A_k ensures that A_k is not affected by a rescaling of the time axis.

Discrete Fourier Series (DFS)

Let $\tilde{s}[n]$ be periodic of period N. For $n \in \mathbb{Z}$ and $k \in \{0, 1, \ldots, N-1\}$,

$$\tilde{s}[n] = \sum_{k=0}^{N-1} B_k e^{j\frac{2\pi}{N}nk}$$

$$B_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{s}[n] e^{-j\frac{2\pi}{N}kn}.$$

Discrete Fourier Transform (DFT)

Let $s[0], \ldots, s[N-1]$ be given. For $n, k \in \{0, \ldots, N-1\}$,

$$s[n] = \frac{1}{N} \sum_{k=0}^{N-1} s_{\mathcal{F}}[k] e^{j\frac{2\pi}{N}nk}$$

$$s_{\mathcal{F}}[k] = \sum_{n=0}^{N-1} s[n] e^{-j\frac{2\pi}{N}kn}.$$

Note the similarity of the DFS and the DFT.

The DFS and DFT are not orthonormal either. An orthonormal version of the DFT is

$$m[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} m_\xi[k] e^{j\frac{2\pi}{N}nk}$$

$$m_\xi[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} m[n] e^{-j\frac{2\pi}{N}nk}$$

For reasons of implementation speed, it is advantageous to leave out the factor $1/\sqrt{N}$. Hence the definition of DFT as it is.

We should mention that we can insert the factor $\frac{1}{N}$ in the transform or in the inverse (but not in both). MATLAB and Python have it in the inverse.

The Main Goal of this Lecture

We want to use MATLAB/Python to determine (or approximate) the Fourier transform of a finite-duration signal $s(t)$.

$s(t)$ is related to its periodic extension $\tilde{s}(t)$ as well as to the sampled versions $\tilde{s}[n]$ and $(s[0], s[1], \ldots, s[N-1])$.

Hence, in principle, the FT of $s(t)$ can be obtained via

- the FS of $\tilde{s}(t)$ or
- the DFS of $\tilde{s}[n]$ or
- the DFT of $(s[0], s[1], \ldots, s[N-1])$.

We choose the latter, since there is a fast algorithm, called the FFT, to compute the DFT.

So, we need to determine the relationship between the **Fourier transform** of a finite-duration signal $s(t)$ and the **DFT** of a vector $(s[0], s[1], \ldots, s[N-1])$ that contains the signal's samples taken every T_s seconds.

We choose $T_p = NT_s$ for some integer N, where N is sufficiently large so that T_p is larger than the signal's duration.

Since the relationship between the Fourier transform of $s(t)$ and that of $s(t - T_{shift})$ for an arbitrary T_{shift} is well understood, we assume that the support of $s(t)$ is contained in $[0, T_p]$.

$s(t)$ **versus** $\tilde{s}(t) := \sum_{i\in\mathbb{Z}} s(t - iT_p)$

Let $\tilde{s}(t) = \sum_{i\in\mathbb{Z}} s(t - iT_p)$ be the periodic extension (period T_p) of $s(t)$.

Clearly we can always go from $s(t)$ to $\tilde{s}(t)$ and back.

For later use, we are interested in the Fourier-domain relationship between $s(t)$ and $\tilde{s}(t)$.

$$
\begin{aligned}
A_k &= \frac{1}{T_p}\int_0^{T_p} \tilde{s}(t) e^{-j\frac{2\pi}{T_p}kt} dt \\
&= \frac{1}{T_p}\int_0^{T_p} \sum_{i\in\mathbb{Z}} s(t - iT_p) e^{-j\frac{2\pi}{T_p}k(t-iT_p)} dt \\
&= \frac{1}{T_p}\int_{-\infty}^{\infty} s(t) e^{-j\frac{2\pi}{T_p}kt} dt \\
&= \frac{1}{T_p} s_{\mathcal{F}}\left(\frac{k}{T_p}\right).
\end{aligned}
$$

Hence the Fourier series coefficients $\{A_k\}_{k\in\mathbb{Z}}$ are obtained by sampling and scaling the Fourier transform $s_{\mathcal{F}}(t)$. The sampling occurs at integer multiples of the frequency $\frac{1}{T_p}$.

We summarize:

$$\begin{array}{ccc} s(t) & \Longleftrightarrow & \tilde{s}(t) \\ \Updownarrow & & \Updownarrow \\ s_{\mathcal{F}}(f) & \Longleftrightarrow & A_k \end{array}$$

with

$$\boxed{\frac{1}{T_p} s_{\mathcal{F}}\left(\frac{k}{T_p}\right) = A_k}$$

(Question: would you know how to determine $s_{\mathcal{F}}(f)$ from $\{A_k\}_{k \in \mathbb{Z}}$?)

$\tilde{s}(t)$ versus $\tilde{s}[n] = \tilde{s}(nT_s)$

We choose $T_p = NT_s$, i.e., we are taking N samples within one period T_p of $\tilde{s}(t)$.

We know that if we satisfy the condition of the sampling theorem, we can reconstruct $\tilde{s}(t)$ from $\tilde{s}(nT_s)$. We want to know how the two relate in the frequency domain.

From the DFS:

$$\tilde{s}[n] = \sum_{k=0}^{N-1} B_k e^{j\frac{2\pi}{N}nk}$$

From the FS:

$$\begin{aligned}
\tilde{s}(nT_s) &= \sum_{k\in\mathbb{Z}} A_k e^{j\frac{2\pi}{T_p}knT_s} \\
&= \sum_{l=0}^{N-1} \sum_{m\in\mathbb{Z}} A_{l+mN} e^{j\frac{2\pi}{N}(l+mN)n} \\
&= \sum_{l=0}^{N-1} \left(\sum_{m\in\mathbb{Z}} A_{l+mN} \right) e^{j\frac{2\pi}{N}ln}
\end{aligned}$$

Now we use the fact that $\tilde{s}[n] = \tilde{s}(nT_s)$. Comparing the corresponding expressions yields

$$\boxed{B_k = \sum_{m\in\mathbb{Z}} A_{k+mN}, \quad k = 0, 1, \ldots, N-1}$$

Clearly we can go from the $\{A_k\}_{k\in\mathbb{Z}}$ to the $\{B_k\}_{k\in\mathbb{Z}}$, which is consistent with the fact that we can always sample a signal.

We can go the other way if the condition of the sampling theorem is met, i.e., if the Fourier transform of $s(t)$ has support contained in an interval of length $1/T_s$. We'll come back to this later.

In the following diagrams, $\overset{(**)}{\Longleftarrow}$ denotes that we can go in the indicated direction whenever the condition of the sampling theorem is fulfilled.

$\tilde{s}[n]$, $n \in \mathbb{Z}$, vs. $s[n], n = 0, \ldots, N-1$

There is a one-to-one map between $\tilde{s}[n]$, $n \in \mathbb{Z}$ and $s[n]$, $n = 0, \ldots, N-1$.

We want to relate the corresponding frequency domain characterizations.

Equating the reconstruction formulas for the DFS and DFT we obtain

$$\boxed{B_k = \frac{1}{N} s_{\mathcal{F}}[k], k = 0, 1, \ldots, N-1}$$

Putting Things Together

Recall that the support of $s(t)$ is contained in $[0, NT_s]$.

$$
\begin{array}{ccccccc}
s(t) & \Longleftrightarrow & \tilde{s}(t) & \overset{(**)}{\overset{\Longleftarrow}{\Longrightarrow}} & \tilde{s}[n] & \Longleftrightarrow & (s[0], \ldots, s[N-1]) \\
\Updownarrow \text{FT} & & \Updownarrow \text{FS} & & \Updownarrow \text{DFS} & & \Updownarrow \text{DFT} \\
s_{\mathcal{F}}(f) & \Longleftrightarrow & \{A_k\}_{k\in\mathbb{Z}} & \overset{(**)}{\overset{\Longleftarrow}{\Longrightarrow}} & \{B_k\}_{k=0}^{N-1} & \Longleftrightarrow & (s_{\mathcal{F}}[0], \ldots, s_{\mathcal{F}}[N-1])
\end{array}
$$

With

$$
s_{\mathcal{F}}[k] = NB_k = N \sum_{l\in\mathbb{Z}} A_{k+lN} = \frac{1}{T_s} \sum_{l\in\mathbb{Z}} s_{\mathcal{F}}\left(\frac{k}{T_p} + \frac{l}{T_s}\right),
$$

where $k = 0, 1, \ldots, N-1$.

From

$$s_{\mathcal{F}}[k] = NB_k = N \sum_{l \in \mathbb{Z}} A_{k+lN} = \frac{1}{T_s} \sum_{l \in \mathbb{Z}} s_{\mathcal{F}}\left(\frac{k}{T_p} + \frac{l}{T_s}\right),$$

it is clear that we can always go from $s_{\mathcal{F}}(f)$ to $s_{\mathcal{F}}[k]$.

How to go the other direction? (It is the other direction that allows us to infer $s_{\mathcal{F}}(f)$ from the MATLAB/Python–computed $s_{\mathcal{F}}[k]$.)

If the support of $s_{\mathcal{F}}(f)$ is contained in an interval $(f_{min}, f_{max}]$ or $[f_{min}, f_{max})$ of length $\frac{1}{T_s}$, then for each $k = 0, 1, \ldots, N-1$, there is exactly one $l = l(k)$ such that $\frac{k}{T_p} + \frac{l}{T_s}$ is in the specified interval.[1]

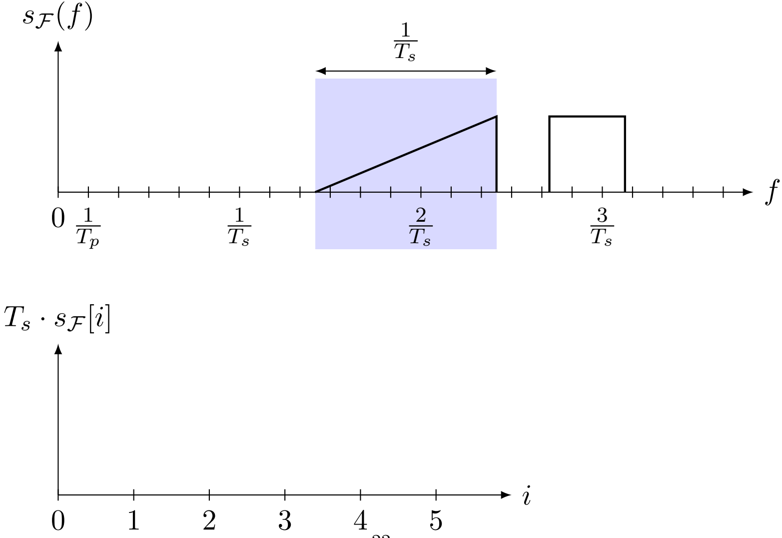
In this case we can write

$$s_{\mathcal{F}}[k] = \frac{1}{T_s} s_{\mathcal{F}}\left(\frac{k}{T_p} + \frac{l(k)}{T_s}\right),$$

$k = 0, 1, \ldots, N-1.$

[1]Strictly speaking, the support of $s_{\mathcal{F}}(f)$ cannot be finite since we have assumed that $s(t)$ is of finite duration. However, for all practical purposes we can say that the support of $s_{\mathcal{F}}(f)$ is contained in some sufficiently large interval.

Example (Passband) with $N = \frac{T_p}{T_s} = 6$



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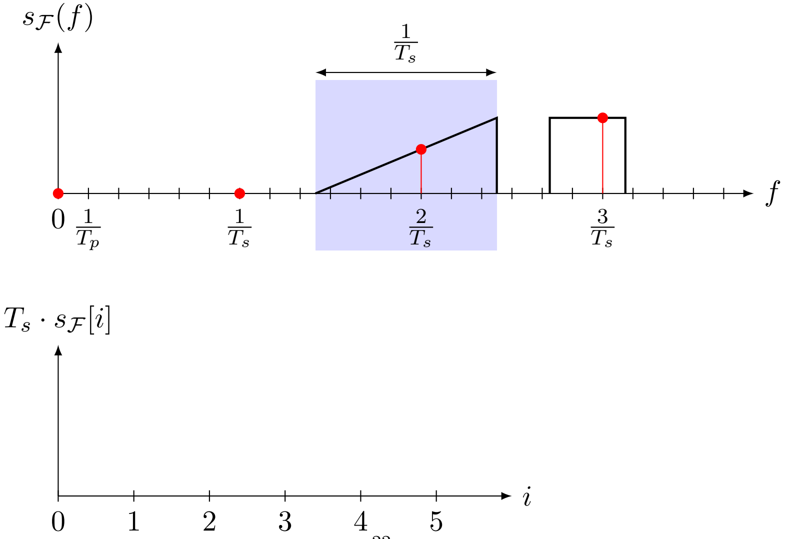

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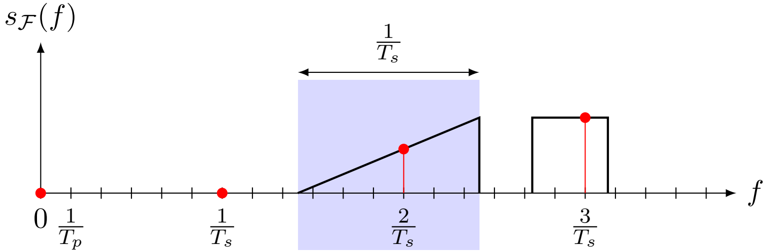

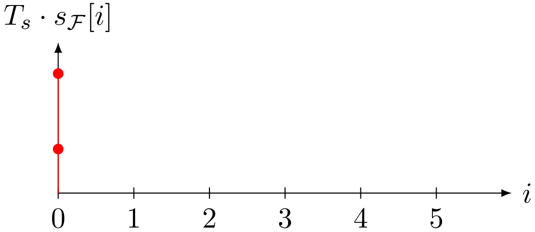

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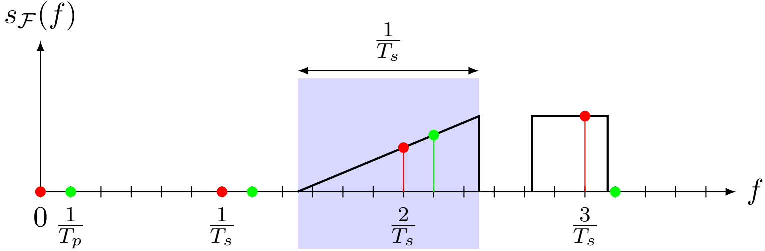

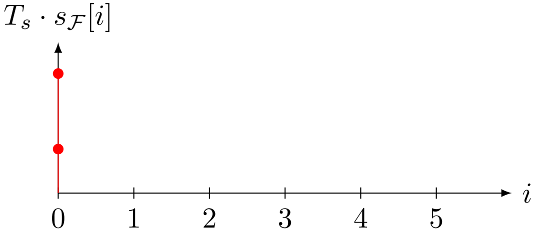

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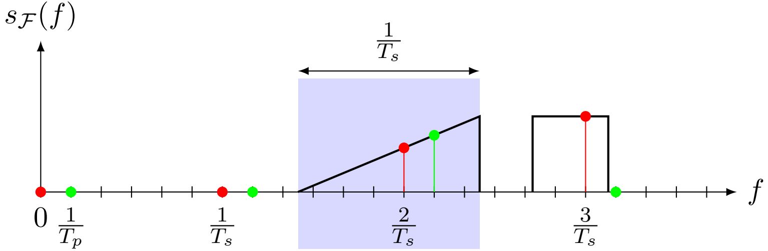

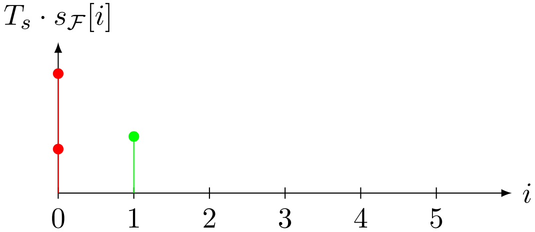

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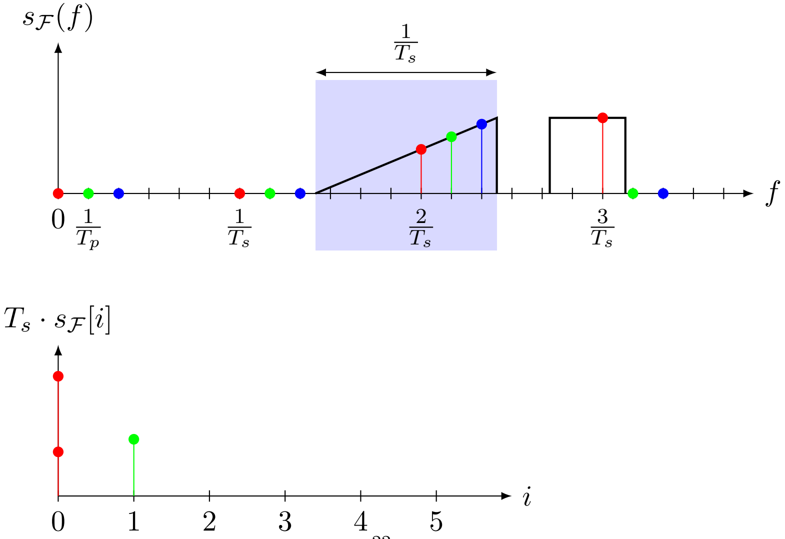

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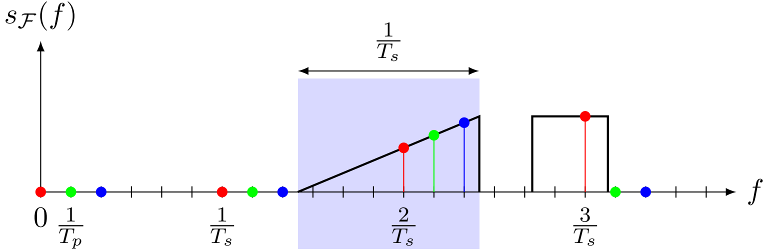

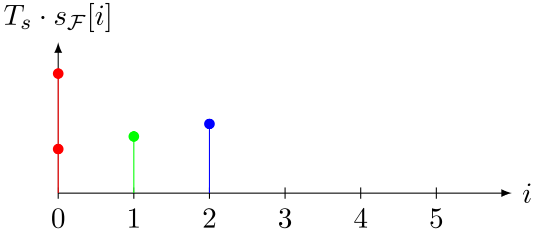

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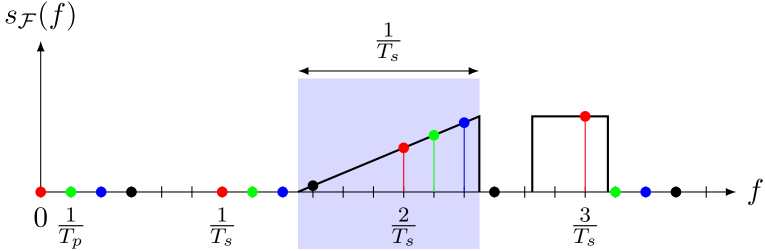

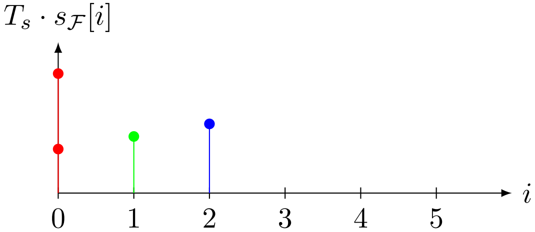

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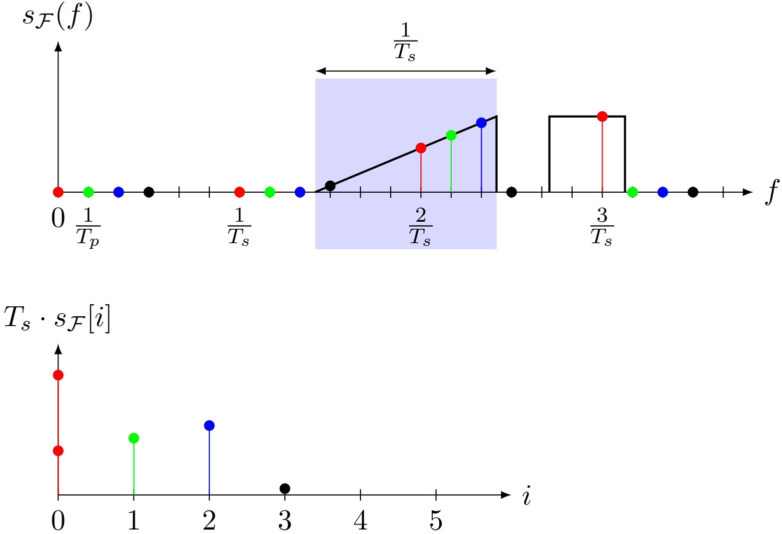

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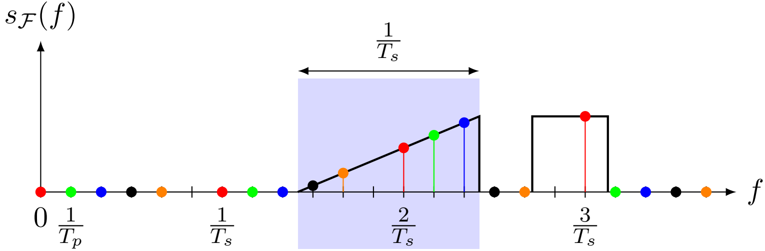

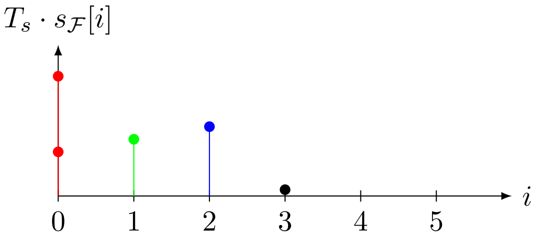

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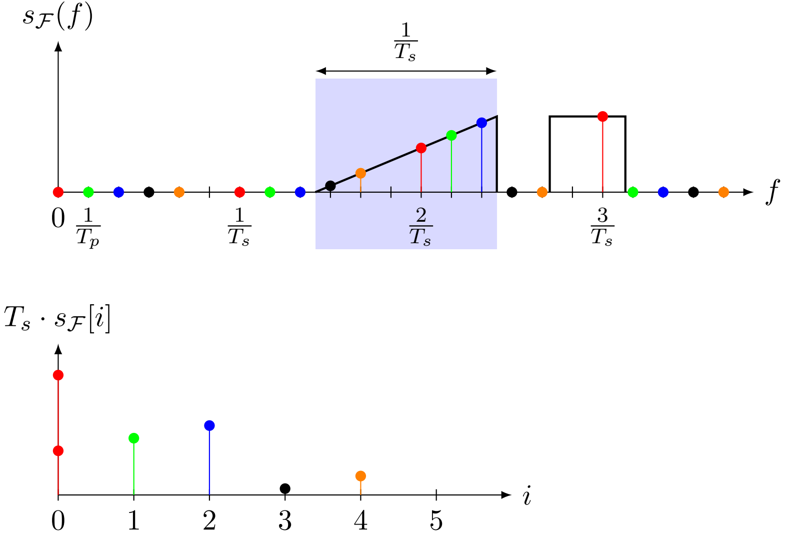

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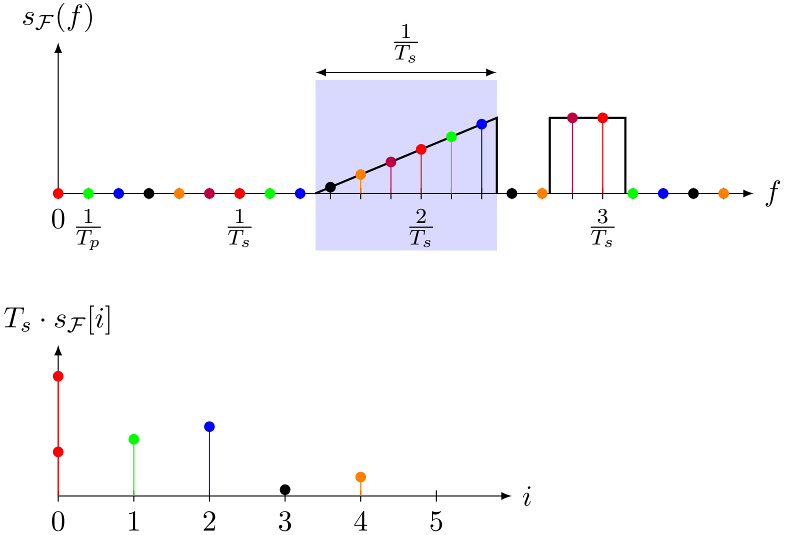

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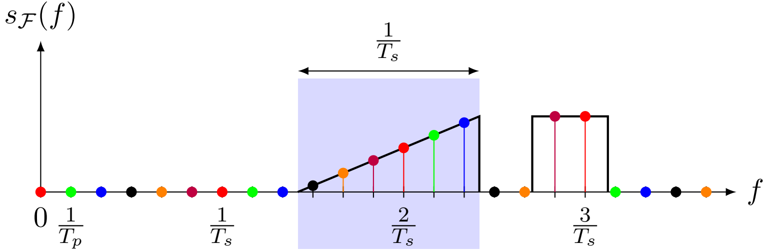
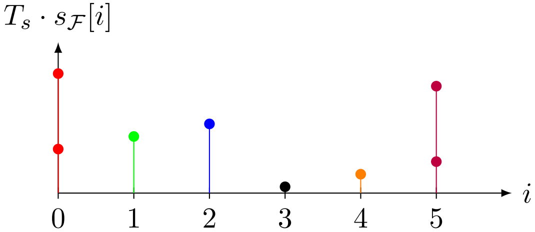




Baseband Signal

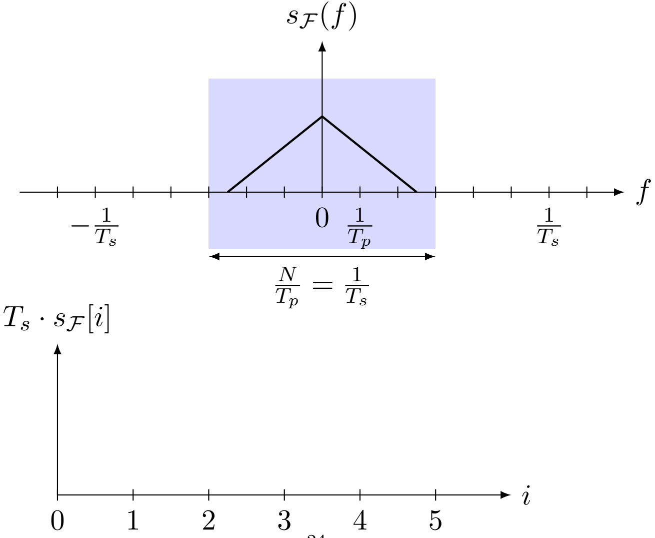
Of particular interest is the case when the support of $s_{\mathcal{F}}(f)$ is contained in $[-\frac{1}{2T_s}, \frac{1}{2T_s})$. (This means that $s(t)$ is a baseband signal).

Recall that

$$s_{\mathcal{F}}[k] = \frac{1}{T_s} \sum_{l \in \mathbb{Z}} s_{\mathcal{F}}\left(\frac{k}{T_p} + \frac{l}{T_s}\right).$$

Hence, as mentioned earlier, since the support of $s_{\mathcal{F}}(f)$ is contained in an interval of length $\frac{1}{T_s}$, then for each $k = 0, 1, \ldots, N-1$, there is exactly one $l = l(k)$ such that $\frac{k}{T_p} + \frac{l}{T_s}$ is in the specified interval. In this case, we can indeed go back from $s_{\mathcal{F}}[k]$ to $s_{\mathcal{F}}(f)$.

Example (Baseband) with $N = \frac{T_p}{T_s} = 6$



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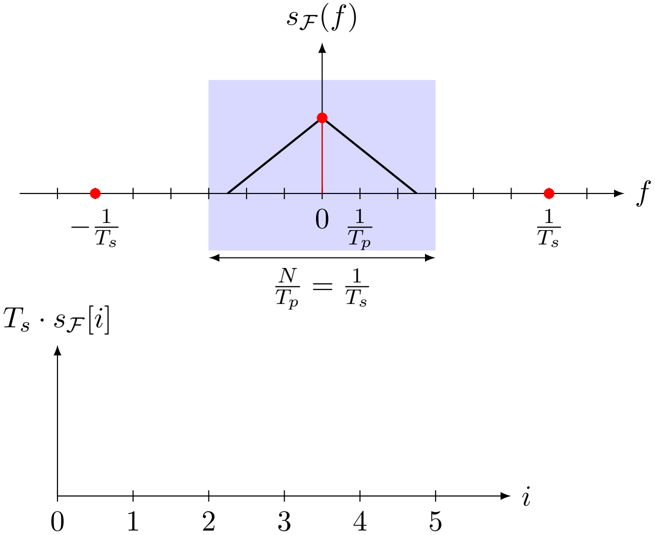

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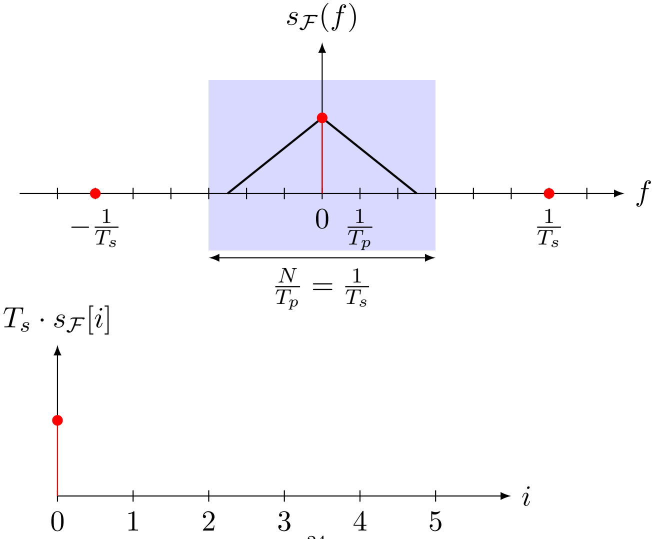

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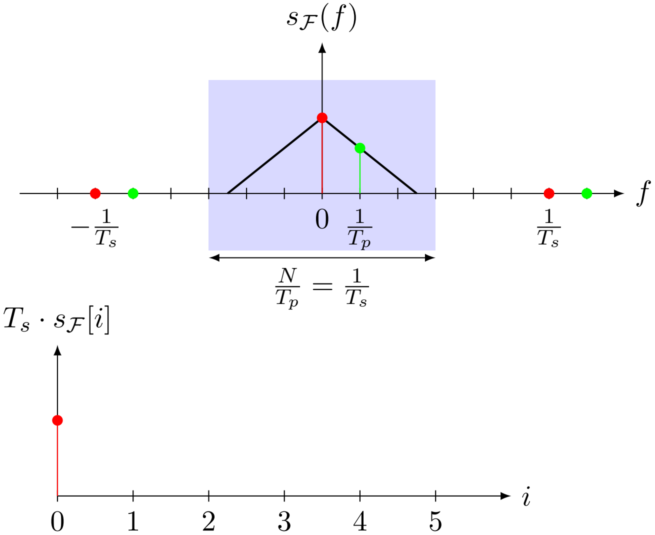

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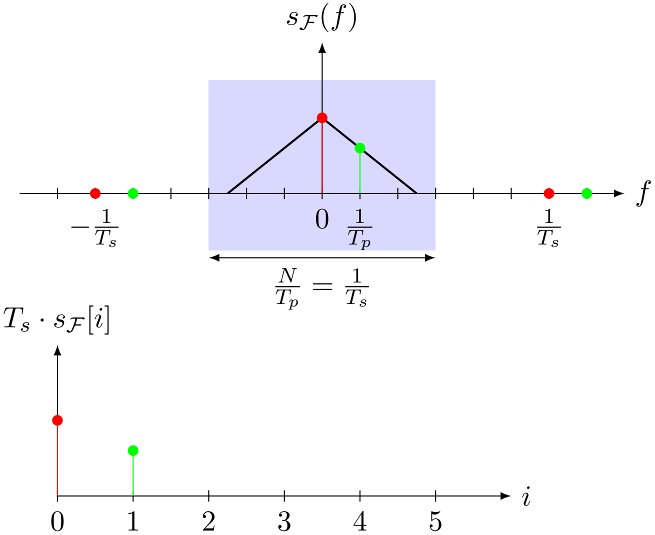

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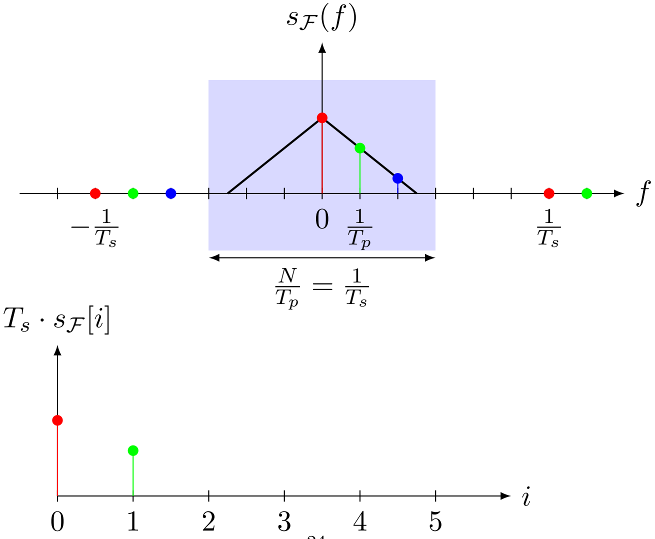

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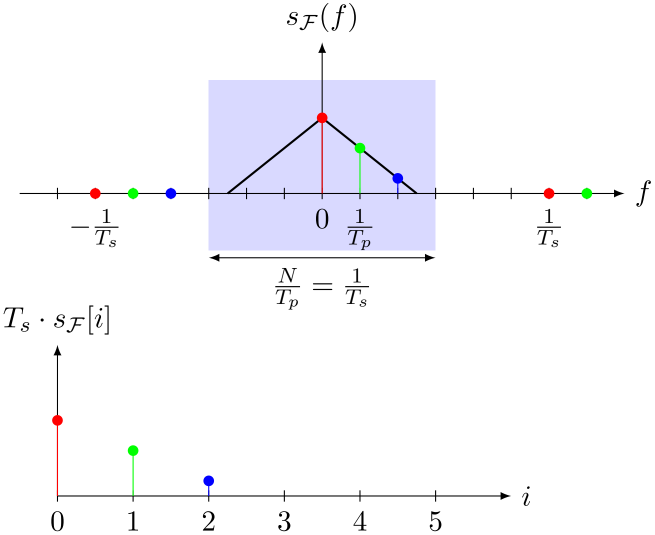

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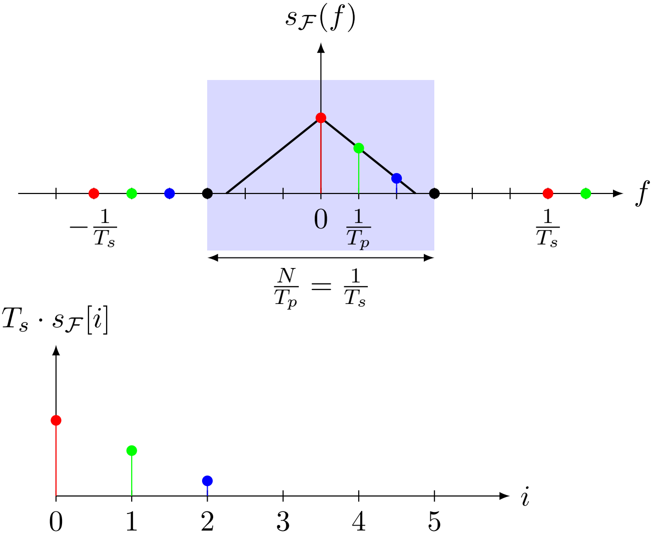

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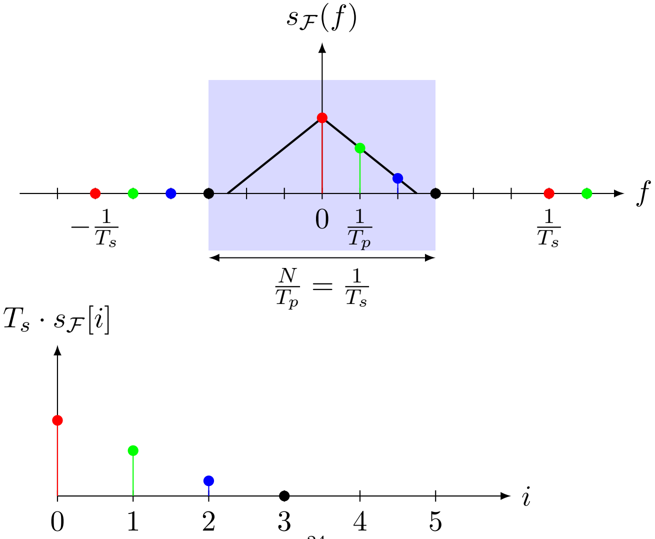

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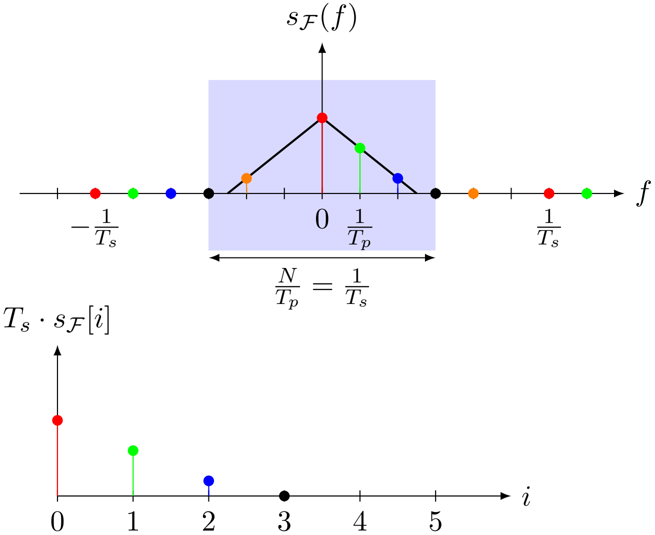

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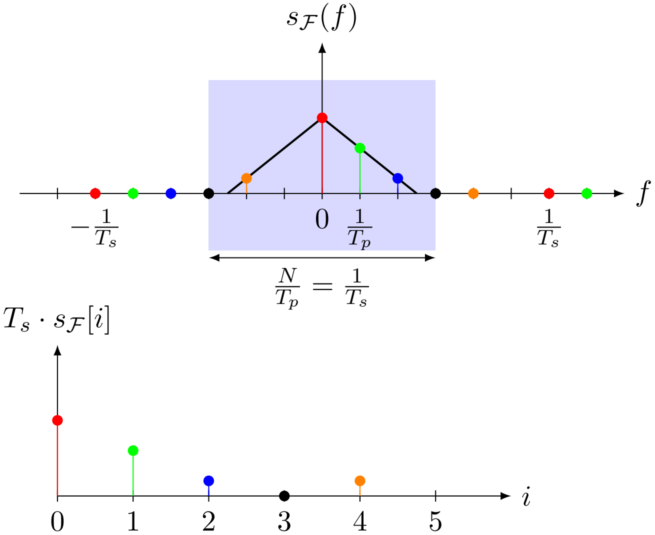

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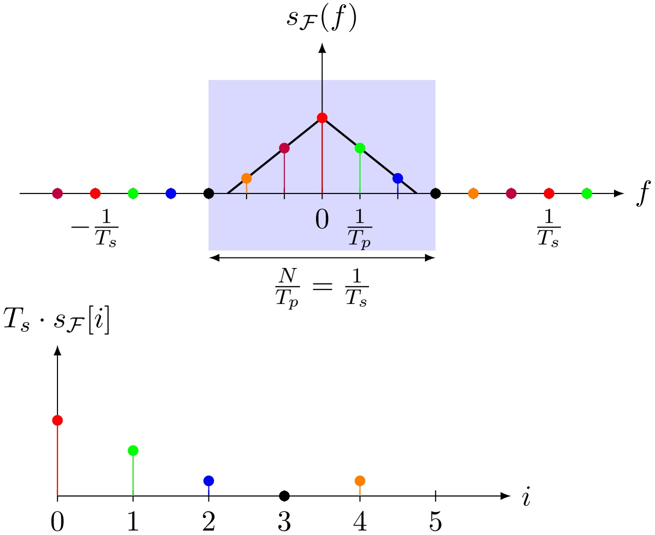
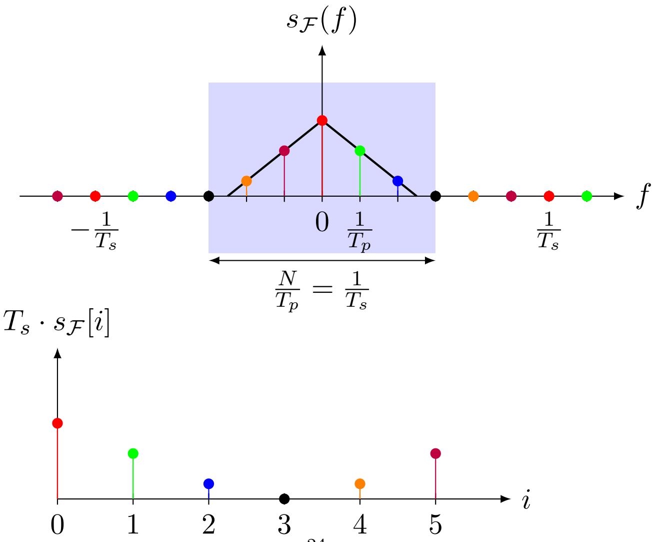


Example (Baseband) with $N = \frac{T_p}{T_s} = 6$



To summarize with the typical situation, suppose that the support of $s_{\mathcal{F}}(f)$ is contained in $[-\frac{1}{2T_s}, \frac{1}{2T_s})$ and N is even, which is always the case to take advantage of the Fast Fourier Transform (FFT).

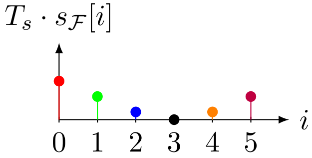
In this case, we have that

$$s_{\mathcal{F}}[k] = \frac{1}{T_s} s_{\mathcal{F}}\left(\frac{k}{T_p} + \frac{l(k)}{T_s}\right),\ \ k = 0, 1, \ldots, N-1.$$

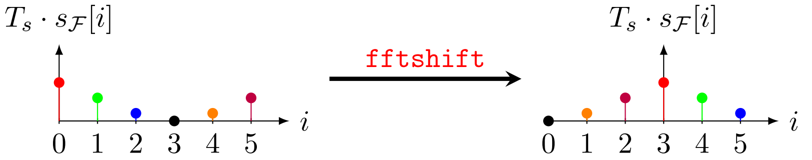
Notice that for $k = N/2$, $\frac{k}{T_p} = \frac{N/2}{T_p} = \frac{1}{2T_s}$ which is already outside the support of $s_{\mathcal{F}}(f)$. So for $k \geq N/2$ we have to choose $l(k) = -1$. (And for $k < N/2$ we choose $l(k) = 0$). Hence the N components of the DFT split into the first $N/2$ components that describe $s_{\mathcal{F}}(f)$ at $N/2$ positive frequencies and the last $N/2$ values that describe it at negative frequencies:

$$s_{\mathcal{F}}[k] = \begin{cases} \frac{1}{T_s} s_{\mathcal{F}}\big(\frac{k}{T_p}\big), & k = 0, 1, \ldots, \frac{N}{2} - 1 \\ \frac{1}{T_s} s_{\mathcal{F}}\big(\frac{k-N}{T_p}\big), & k = \frac{N}{2}, \ldots, N-1. \end{cases}$$

The `fftshift` operation



The `fftshift` operation



After the `fftshift` operation, the corresponding frequency vector is

$$\left(\underbrace{\frac{-N/2}{T_p}, \frac{-N/2+1}{T_p}, \ldots, \frac{-1}{T_p}}_{N/2 \text{ components}}, \underbrace{0, \frac{1}{T_p}, \ldots, \frac{N/2-1}{T_p}}_{N/2 \text{ components}}\right).$$

Or equivalently

$$\left(\underbrace{-\frac{1}{2T_s}, -\frac{1}{2T_s} + \frac{1}{T_p}, \ldots, \frac{-1}{T_p}}_{N/2 \text{ components}}, \underbrace{0, \frac{1}{T_p}, \ldots, \frac{1}{2T_s} - \frac{1}{T_p}}_{N/2 \text{ components}}\right).$$

(In Python: `numpy.fft.fftshift`)