Homework 12

Exercise 1. Let X, Y be two discrete random variables (with values in a countable set C). Let us moreover assume that X is integrable.

a) Show that the random variable $\psi(Y)$, where ψ is defined as

$$\psi(y) = \sum_{x \in C} x \, \mathbb{P}(\{X = x\} | \{Y = y\})$$

matches the definition of conditional expectation $\mathbb{E}(X|Y)$ given in the lectures.

b) *Application:* One rolls two independent and balanced dice (say Y and Z), each with four faces. What is the conditional expectation of the maximum of the two, given the value of one of them?

Exercise 2*. *Application (Zombie Apocalypse).* A zombie bites N people during its lifetime (undeath time?), where N has the Poisson distribution with parameter λ. Once a person is bitten, they become a zombie with probability p or their brain is eaten with probability $q = 1-p$, independently of all other bitten individuals. Let Z be the number of zombie offsprings created by a zombie.

a) Find $\mathbb{E}(Z|N)$ and $\mathbb{E}(Z)$

b) Find $\mathbb{E}(N|Z)$

c) *Optional (not graded)*: Assume that the initial zombie infestation is small (relative to the population of the planet). According to the above model, for which values of p and λ would you expect the infestation to turn into a full blown apocalypse and for which values would it die down locally?

Exercise 3. Let X be a random variable such that $\mathbb{P}(\{X = +1\}) = \mathbb{P}(\{X = -1\}) = \frac{1}{2}$ and $Z \sim \mathcal{N}(0, 1)$ be independent of X. Let also $a > 0$ and $Y = aX + Z$. We propose below four possible estimators of the variable X given the noisy observation Y:

$$\widehat{X}_1 = \frac{Y}{a} \qquad \widehat{X}_2 = \frac{aY}{a^2+1} \qquad \widehat{X}_3 = \text{sign}(aY) \qquad \widehat{X}_4 = \tanh(aY)$$

a) Which estimator among these four minimizes the mean square error (MSE) $\mathbb{E}((\widehat{X} - X)^2)$?

In order to answer the question, draw on the same graph the four curves representing the MSE as a function of $a > 0$. For this, you may use either the exact mathematical expression of the MSE or the one obtained via Monte-Carlo simulations.

b) Provide a theoretical justification for your conclusion.

c) For which of the four estimators above does it hold that $\mathbb{E}((\widehat{X} - X)^2) = \mathbb{E}(X^2) - \mathbb{E}(\widehat{X}^2)$?

Exercise 4.

a) Let $(M_n, n \in \mathbb{N})$ be an *increasing* martingale, that is, $M_{n+1} \geq M_n$ a.s. for all $n \in \mathbb{N}$. Show that $M_n = M_0$ a.s., for all $n \in \mathbb{N}$.

b) Let $(M_n, n \in \mathbb{N})$ be a square-integrable martingale such that $(M_n^2, n \in \mathbb{N})$ is also a martingale. Show that $M_n = M_0$ a.s., for all $n \in \mathbb{N}$.