ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
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Handout 35 Information Theory and Coding
Final exam January 23, 2021

4 problems, 92 points
165 minutes
Good Luck!

PROBLEM 1. (19 points) Let W_1 and W_2 be binary input channels. Consider a scheme that takes i.i.d. binary inputs U_1, U_2 with $\Pr(U_i = 0) = \Pr(U_i = 1) = 1/2$, and produces $X_1 = U_1 \oplus U_2$ and $X_2 = U_2$. We transmit X_1 through channel W_1, and X_2 through channel W_2. W_1 and W_2 operate on X_1 and X_2 independently. Let Y_1 be the output of channel W_1, and Y_2 be the output of channel W_2.

Under this scheme, define W^- as the channel with U_1 as its input and Y_1, Y_2 as its output. Also define W^+ as the channel with U_2 as its input and Y_1, Y_2, U_1 as its output.

Recall from the lectures that $I(W^-) := I(U_1; Y_1, Y_2)$, $I(W^+) := I(U_2; Y_1, Y_2, U_1)$ and $I(W_i) := I(X_i; Y_i)$ for $i = 1, 2$.

a. Show that $I(W^-) + I(W^+) = I(W_1) + I(W_2)$.

b. Show that $I(W^+) \geq I(W_2)$ and $I(W^-) \leq I(W_1)$.

c. Show that $I(W^-) \leq \min(I(W_1), I(W_2))$ and $I(W^+) \geq \max(I(W_1), I(W_2))$

   Hint: what would happen to the value of $I(W^-)$ if we exchange W_1 and W_2?

d. Let W_1 be $BEC(\epsilon)$, and W_2 be $BSC(p)$, determine the value of $I(W^+)$ and $I(W^-)$.

PROBLEM 2. (23 points) Given random variables X and Y, define

$$K(X;Y) = \min_{p_{U|XY}: X-U-Y} I(XY;U),$$

i.e., the minimum value of $I(XY;U)$ among all possible distributions $p_{U|XY}$ such that $X-U-Y$ is a Markov chain. Note that X and Y are not necessarily independent.

a. Show that $K(X;Y) \geq I(X;Y)$.

   Hint: It suffices to show that $X-U-Y$ implies $I(X;Y) \leq I(XY;U)$

b. Suppose $p_{U|XY}$ is such that $X-U-Y$. Suppose further that there exists $u_1, u_2 \in \mathcal{U}, u_1 \neq u_2$, such that $p_{X|U}(.|u_1) = p_{X|U}(.|u_2)$. Show that there exists $p_{V|XY}$ such that $X-V-Y$, $|\mathcal{V}| < |\mathcal{U}|$ and $I(XY;V) \leq I(XY;U)$.

c. Show that among the minimizers there is a U such that whenever $u_1 \neq u_2$ either $p_{X|U}(.|u_1) \neq p_{X|U}(.|u_2)$ or $p_{Y|U}(.|u_1) \neq p_{Y|U}(.|u_2)$.

Suppose now that X and Y are binary with $p_{XY}(00) = p_{XY}(11) = p_{XY}(01) = 1/3$, $p_{XY}(10) = 0$.

d. Show that the minimum for the XY as above is achieved by a binary valued U such that $p_{X|U}(x|0) = \mathbb{1}\{x=0\}$ and $p_{Y|U}(y|1) = \mathbb{1}\{y=1\}$.

e. Show that the minimum for the XY as above is achieved by a binary U with $\Pr(U = 0) = \Pr(U=1) = 1/2$, and $K(X;Y) = 2/3$.

PROBLEM 3. (26 points) Let $X_1, X_2, \ldots$ be a Gaussian process (i.e., X^n is a Gaussian random vector for each n.). Assume that X^n has a PDF, i.e., X^n has a non-degenerate multivariate Gaussian distribution. Let f_n denote this PDF. Let K_n denote the covariance matrix of X^n; such a matrix K_n can be uniquely factorized as $B_n B_n^T$ where B_n is lower triangular with positive diagonal entries.

a. Show that there exists constants $\{a_{ij} : 1 \leq j \leq i\}$ and $\{m_i : i \geq 1\}$ so that

$$Z_i = \sum_{j=1}^{i} a_{ij}(X_j - m_j), i = 1, 2, \ldots$$

are i.i.d. $N(0, 1)$.

b. Show that

$$-\frac{1}{n} \log f_n(X^n) - \frac{1}{n} h(X^n) = \frac{\log(e)}{2n} \sum_{i=1}^{n} (Z_i^2 - 1).$$

c. Show that

$$\lim_{n\to\infty} \left[-\frac{1}{n} \log f_n(X^n) - \frac{1}{n} h(X^n) \right] = 0$$

with probability 1.

Now consider the process $X_i = X_{i-1} + iZ_i$, where $X_0 = 0$ and Z_i are i.i.d. $N(0, 1)$.

d. Is the process above stationary?

e. Find if $\lim_{n\to\infty} h(X^n)/n$ exists.

f. Does part c. hold for this process?

Problem 4. (24 points) Suppose $U_1, U_2, \dots$ are i.i.d. binary random variables with $\Pr(U = 1) = \Pr(U = 0) = 1/2$. Suppose $V_1, V_2, \dots$ are obtained by passing $U_1, U_2, \dots$ through a memoryless binary erasure channel with erasure probability p. The sequence $U_1, U_2, \dots$ is observed by Alice, the sequence $V_1, V_2, \dots$ is observed by Bob.

Alice wishes to describe U^n to Bob by sending him a message W_n (which is a function of U^n) over a noiseless channel. Bob is then supposed to form an estimate $\hat{U}^n$ of U^n from (W_n, V^n).

Let $q_n = \frac{1}{n}\sum_{i=1}^{n} \Pr(\hat{U}_i \neq U_i)$ denote the 'bit error probability' of Bob's estimate.

a. Show that $\frac{1}{n}H(U^n|\hat{U}^n) \leq h_2(q_n)$.

b. Show that $\frac{1}{n}I(U^n;\hat{U}^n) \leq (1-p) + \frac{1}{n}H(W_n)$.

c. With $\mathcal{W}_n$ denoting the set of possible values of W_n, show that

$$\frac{1}{n}\log|\mathcal{W}_n| \geq p - h_2(q_n).$$

Fix $R \geq 0$. For each u^n in $\{0,1\}^n$, assign a label $W_n(u^n)$ chosen uniformly at random from $\{1, \dots, 2^{nR}\}$. These labels are chosen independently, i.e., $W_n(u^n)$ are independent of $W_n(u'^n)$ if $u'^n \neq u^n$. Reveal these labels to both Alice and Bob.

d. Given V^n and $W_n(U^n)$, how should Bob choose $\hat{U}^n$ so that $\Pr(\hat{U}^n \neq U^n)$ is minimized?

e. Now fix R with $R > p$. Show that with the optimal method in d., there exists a sequence of labels $W_n(.)$ such that

$$\lim_{n\to\infty} \Pr(\hat{U}^n \neq U^n) = 0.$$

Hint: Letting K denote the number of erasures in V^n, find an upper bound for $\Pr(\hat{U}^n \neq U^n|K = k)$. Then pick r in (p, R), and separately consider the cases $K > nr$ and $K \leq nr$.