Information Theory & Coding

Nov 10th, 2020

Yesterday

$$x \to x + Z \qquad Z \sim N(0, \sigma^2)$$

$b(x) = x^2$, we saw

$$C(P) = \frac{1}{2}\log\left(1 + P/\sigma^2\right)$$

Today: "Additive Gaussian Noise Waveform Channel"

$$x(t) \longrightarrow Y(t) = x(t) + Z(t).$$

input $\qquad$ output $\qquad t \in \mathbb{R}$

the process $(Z(t) : t \in \mathbb{R})$ has statistics independent of $(x(t) : t \in \mathbb{R})$

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what is $Z(t)$, the noise?

when we measure Z at time t_0, we in fact mean a measurement of the form

$$\int_{t_0 - \Delta/2}^{t_0 + \Delta/2} Z(t)\, dt$$

$$Z_1 = \underbrace{\int_a^{a+\Delta} z(t)\,dt} \quad Z_2 = \underbrace{\int_b^{b+\Delta} z(t)\,dt}$$

if a & b are far apart then we expect Z_1 & Z_2 to be independent.

An idealized version will say if

$$(\underbrace{a, a+\Delta}) \cap (\underbrace{b, b+\Delta}) = \phi \Rightarrow Z_1 \,\&\, Z_2 \text{ indep. \& identically distrib.}$$

It will also assume that Z_1 & Z_2 have zero-mean.

$$Z_3 = \underbrace{\int_a^{a+2\Delta} z(t)\,dt}_{\substack{0\text{-mean} \\ \text{variance } f(2\Delta)}} = \underbrace{\underbrace{\int_a^{a+\Delta} z(t)\,dt}_{0\text{-mean}} + \underbrace{\int_{a+\Delta}^{a+2\Delta} z(t)\,dt}_{\substack{0\text{-mean} \\ \text{variance } f(\Delta)}}}_{\text{same variance } f(\Delta)}$$

0-mean variance $2f(\Delta)$

$$\Rightarrow f(\Delta) = \sigma^2 \Delta.$$

$$Z_b = \int g(t)\, z(t)\,dt \approx \sum_n \int_{n\Delta}^{(n+1)\Delta} g(t)\, z(t)\,dt$$

$$\approx \sum_n \int_{n\Delta}^{(n+1)\Delta} g(n\Delta)\, z(t)\,dt$$

$$\int_{n\Delta}^{(n+1)\Delta} g(n\Delta) Z(t)\,dt = g(n\Delta) \underbrace{\int_{n\Delta}^{(n+1)\Delta} Z(t)\,dt}_{\text{0-mean variance } \sigma^2\Delta}$$

0-mean variance $g(n\Delta)^2 \sigma^2 \Delta$.

$\int Z(t) g(t)\,dt$ is 0-mean variance

$\sigma^2 \sum_n g(n\Delta)^2 \Delta \approx \sigma^2 \int g(t)^2\,dt$.

$$Z_1 = \int g_1(t)\, Z(t)\, dt$$

$$\simeq \sum_n g_1(n\Delta) \int_{n\Delta}^{(n+1)\Delta} Z(t)\, dt$$

$$Z_2 = \int g_2(t)\, Z(t)\, dt$$

$$\simeq \sum_m g_2(m\Delta) \int_{m\Delta}^{(m+1)\Delta} Z(t)\, dt$$

$$E(Z_1 Z_2) = \sum_{n,m} g_1(n\Delta)\, g_2(m\Delta)\, E\left[\int_{n\Delta}^{(n+1)\Delta} Z(t)\, dt \int_{m\Delta}^{(m+1)\Delta} Z(t)\, dt\right]$$

$$= \begin{cases} 0 & \text{if } n \neq m \\ \sigma^2 \Delta & \text{if } n = m \end{cases}$$

$$= \sum_n g_1(n\Delta)\, g_2(n\Delta)\, \sigma^2 \Delta$$

$$\simeq \sigma^2 \int g_1(t)\, g_2(t)\, dt.$$

Z is an agglomeration of microscopic phenomena.

$$X = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i \sqrt{n})$$

X_i 0-mean, indep.

$E(X_i^2) \simeq O\left(\frac{1}{n}\right)$

Central limit theorem will say X is Gaussian in the limit of large n.

The "idealized" noise process $Z(t)$ is characterized by the following property: whenever $g_1, g_2, \ldots, g_n$ are functions of time and we form

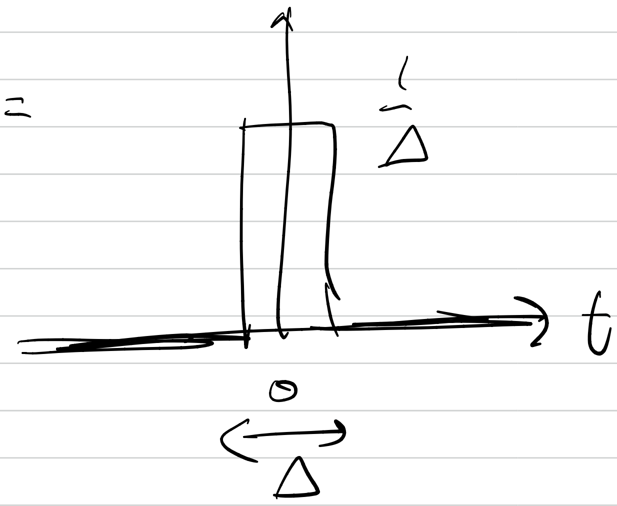
$$Z_1 = \int g_1(t) Z(t)\,dt, \ldots, Z_n = \int g_n(t) Z(t)\,dt$$

then $\begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$ is a Gaussian Random Vector with 0-mean & covariance matrix

K with $K_{ij} = \sigma^2 \int g_i(t) g_j(t)\,dt$.

In the literature such a $(Z(t) : t \in \mathbb{R})$ is called "white" gaussian noise of intensity σ^2.

$g_1(t) =$



had $Z(t)$ been a non-random, continuous then

$$\lim_{\Delta \to 0} \int g_1(t) Z(t)\,dt = Z(0).$$

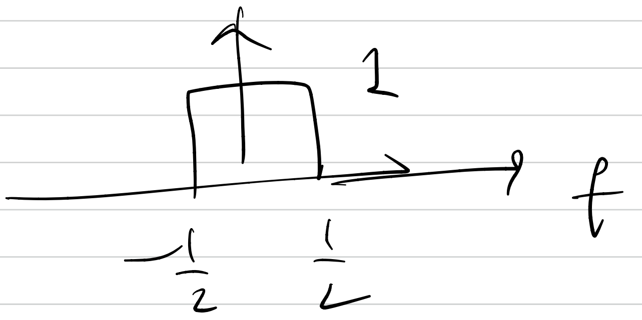
$$\int g_1(t) Z(t)\,dt$$

$$= \text{Gaussian}\left(0\text{-mean}, \text{variance} = \frac{\sigma^2}{\Delta}\right)^*$$

$g_1(t) = \mathrm{sinc}(t)$

$$\mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t}$$

$\Updownarrow$ Fourier Transform



$$\int x(t)\, e^{-j2\pi f t}\, dt = X(f)$$

$$\int X(f)\, e^{+j2\pi f t}\, df = x(t)$$

Parseval Formula : $\displaystyle\int x(t)\, y^*(t)\, dt$

$$= \int X(f)\, Y^*(f)\, df$$

$g_1(t) = \mathrm{sinc}(t)$

$g_2(t) = \mathrm{sinc}(t-1)$

$z_1 = \int z(t)\, g_1(t)\, dt$

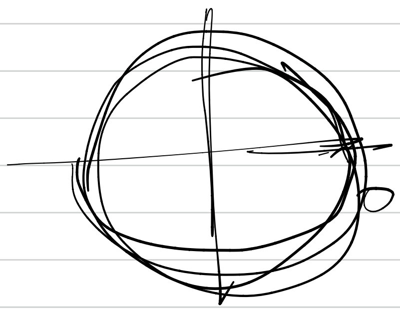
$z_2 = \int z(t)\, g_2(t)\, dt$

$E(z_1^2) = \sigma^2 \int g_1(t)^2 dt = \sigma^2 \int_{-1/2}^{1/2} 1^2 \, df = \sigma^2$

$E(z_2^2) = \sigma^2$

$E(z_1 z_2) = \sigma^2 \int \mathrm{sinc}(t)\, \mathrm{sinc}(t-1)\, dt$

$= \sigma^2 \int_{-1/2}^{1/2} 1 \cdot e^{-j2\pi f} \, df = 0.$



So z_1, z_2 are iid $N(0, \sigma^2)$.

$g_i(t) = \mathrm{sinc}(t-i) \quad i \in \mathbb{Z}$

$z_i = \int z(t)\, g_i(t)\, dt$

we will find $(z_i : i \in \mathbb{Z})$ are iid $N(0, \sigma^2)$.

Suppose we have such a $Z(t)$ and we have a channel

$$x(t) \longrightarrow \underline{Y(t)} = x(t) + \underline{Z(t)}.$$

Suppose $x(t)$ belongs to a finite dimensional linear space, i.e, $\exists$ basis functions

~~$X(t) =$~~ $\varphi_1(t), \ldots, \varphi_n(t)$ s.t

$$\boxed{x(t) = \sum_{i=1}^{n} x_i \varphi_i(t).}$$ we might assume

that φ_i's are orthonormal.

$$\int \varphi_i(t) \varphi_j(t) \, dt = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Example: suppose $x(t) \in \{ x_1(t), \dots, x_M(t) \}$

in this case $x(t)$ belongs to the space spanned

by $x_1(t), \dots, x_M(t)$

$$= \{ \bar{x}(t) : \bar{x}(t) = \sum_{i=1}^{M} \alpha_i x_i(t) \quad \alpha_i \in \mathbb{R} \}$$

$(x(t))$ is specified by specifying $(x_1, \dots, x_n)$

$$\left(x(t) = \sum_i x_i \varphi_i(t) \right)$$

$$x_i = \int x(t) \varphi_i(t) \, dt$$

So the input $x(t)$ to the channel is

in ~~fact~~ $\equiv$ to $(x_1, \dots, x_n)$.

Let us now consider the output $(Y(t) : t \in \mathbb{R})$.

$$= (x(t) + Z(t) : t \in \mathbb{R}).$$

Let us form the "measurements"

$$Y_i = \int Y(t)\, \varphi_i(t)\, dt \qquad i=1 \dots, n.$$

$$= \underbrace{\int x(t)\, \varphi_i(t)\, dt}_{x_i} + \underbrace{\int z(t)\, \varphi_i(t)\, dt}_{z_i}$$

$$z_1, z_2 \dots \quad \text{iid } N(0, \sigma^2).$$

Any additional measurements we can do will be

$$\tilde{Y}_i = \int Y(t)\, g_i(t)\, dt \qquad i=1 \dots, m$$

let $g_i(t) = \sum_{j=1} \alpha_{ij} \varphi_j(t) + \tilde{g}_i(t)$

with

$$\alpha_{ij} = \int g_i(t)\, \varphi_j(t)\, dt$$

$$\Rightarrow \int \tilde{g}_i(t)\, \varphi_j(t)\, dt = 0 \qquad \forall j = 1 \dots n.$$

$$\tilde{Y}_i = \sum_j \alpha_{ij} Y_j + \int (z(t) + x(t))\, \tilde{g}_i(t)\, dt$$

$$= \sum_j \alpha_{ij} Y_j + \underbrace{\int z(t)\, \tilde{g}_i(t)\, dt}_{\tilde{z}_i}$$

So the combined measurements

$$\underbrace{Y_1 \dots Y_n, \tilde{Y}_1 \dots \tilde{Y}_m}$$

$$\equiv \underbrace{(Y_1 \dots Y_n, \tilde{Z}_1 \dots \tilde{Z}_m)}$$

$$= (x_1 + Z_1 \dots x_n + Z_n, \underbrace{\tilde{Z}_1, \dots \tilde{Z}_m})$$

with $(\tilde{Z})$ independent of $(Z_1 \dots Z_n)$.

$$\Rightarrow \underbrace{I(X_1 \dots X_n ; \underbrace{Y_1 \dots Y_n} \underbrace{\tilde{Y}_1 \dots \tilde{Y}_m})}$$

$$= I(X^n ; \cancel{X}^n + Z^n, \tilde{Z}^m)$$

with $X^n, Z^n, \tilde{Z}^m$ independents

$$= I(X^n ; X^n + Z^n) + \underbrace{I(X^n ; \tilde{Z}^m | X^n + Z^n)}$$

$$h(\tilde{Z}^m | X^n + Z^n) - h(\tilde{Z}^m | X^n, X^n + Z^n)$$

$$h(\tilde{Z}^m) - h(\tilde{Z}^m)$$

$$= I(X^n ; Y^n).$$

Conclusion Y^n is sufficient. The additional measurements $\tilde{Y}^n$ are unnecessary.

So the channel between

$$\underline{x(t)} \quad \& \quad \underline{Y(t)} = x(t) + Z(t)$$

$$\equiv \underline{(X_1 \ldots X_n)} \quad \& \quad (Y_1 \ldots Y_n) =$$

$$\underline{(X_1 \ldots X_n) + (Z_1 \ldots Z_n)}$$

iid $N(0, \sigma^2)$.

$$(Y_i = X_i + Z_i)$$

Suppose now we have the waveform channel

$$x(t) \rightsquigarrow x(t) + Z(t).$$

$x(t)$ is bandlimited.

$$\frac{1}{2T}\int_{-T}^{T} x(t)^2 \, dt \le P$$

$X(f)$

Recall the sampling thm.

$$(x(t) : t \in \mathbb{R}) \longleftrightarrow (x_n : n \in \mathbb{Z})$$

$$x_n = x\left(\frac{n}{2W}\right)$$

$$x(t) = \left(\sum_n{}' \frac{x_n}{\sqrt{2W}} \boxed{\operatorname{sinc}(2Wt-n)\sqrt{2W}} \right)$$

$$x_n = x\left(\frac{n}{2W}\right) = \sqrt{2W} \int x(t) \operatorname{sinc}(2Wt-n) \sqrt{2W}\, dt.$$

Equivalently

$$x(t) = \sum_n{}' x_n \psi_n(t)$$

$$x_n = x\left(\frac{t}{2W}\right)\sqrt{2W} = \int x(t)\psi_n(t)\,dt$$

$$\psi_n(t) = \sqrt{2W}\cdot \operatorname{sinc}(2Wt-n)$$

$$\int \psi_i(t)\psi_j(t)\,dt = \begin{cases} 1 & i=j \\ 0 & i\neq j. \end{cases}$$

the channel between $(x(t) : t \in \mathbb{R})$
$(Y(t) : t \in \mathbb{R})$

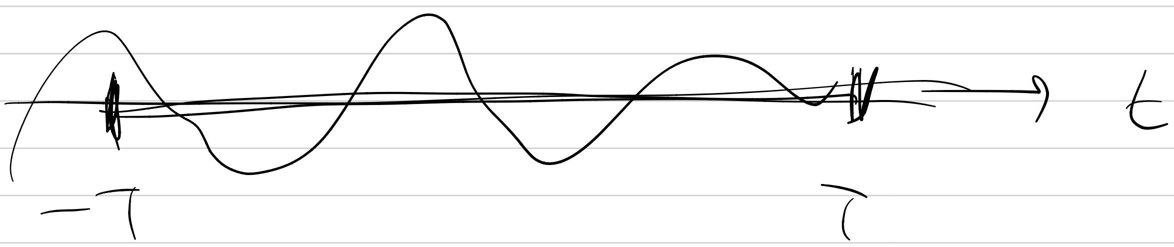
$\equiv$ between $(x_i : i \in \mathbb{Z})$ and $(Y_i : i \in \mathbb{Z})$.

$$Y_i = x_i + Z_i \qquad Z_i \sim N(0, \sigma^2)$$

$$= \int Y(t)\psi_i(t)\,dt$$

$$x(t) = \sum_i{}' x_i \varphi_i(t)$$

$$= \sum_i{}' x_i \operatorname{sinc}(2Wt - i)\sqrt{2W}$$



for $t \in (-T, T]$

$$\approx \sum_{i=-2WT}^{+2WT}{}' x_i \psi_i(t)$$

$$\Rightarrow \frac{1}{2T}\int_{-T}^{T} x(t)^2\, dt \approx \frac{1}{2T}\sum_{i=-2WT}^{+2WT}{}' x_i^2 \approx 2W \frac{1}{4WT}\sum_{i=-2WT}^{+2WT}{}' x_i^2$$

to satisfy the power constraint

$\frac{1}{2T}\int_{-T}^{T} x(t)^2\, dt \le P$ we need the

$(x_i : i \in \mathbb{Z})$ to satisfy

$$\frac{1}{2N}\sum_{i=-N}^{N} x_i^2 \le \frac{P}{2W}$$

So need to consider the capacity of the discrete time channel

$$x_i \longrightarrow Y_i = x_i + Z_i$$

$N(0, \sigma^2)$

obeys the power constraint
$E(X_i^2) \le P/2W$

$$\Rightarrow \quad C = \frac{1}{2} \log \left(1 + \frac{P}{2W\sigma^2}\right) \quad \text{bits/sample}$$

(in discrete time)

in each second we use the channel $2W$ times, so

$$C = W \log \left(1 + \frac{P}{2W\sigma^2}\right) \quad \text{bits/sec}$$

with $\sigma^2 = \frac{N_0}{2}$ the intensity of the white Gaussian n.n

we get

$$\boxed{C = W \log \left(1 + \frac{P}{N_0 W}\right)} \quad \text{bits/sec}$$

as the capacity of the AWGN channel with white-noise intensity $N_0/2$, input power $= P$ & input bandwidth W.