ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 18 — Principles of Digital Communications
Midterm exam — Apr. 13, 2022

4 problems, 165 minutes.
1 sheet (2 pages) of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

PROBLEM 1. Consider the following conditional probability distribution,

$$P_{Y|H}(y \,|\, -1) = \begin{cases} 0.2 & y = 2 \\ 0.2 & y = 1 \\ 0.3 & y = -1 \\ 0.3 & y = -2 \end{cases} \qquad P_{Y|H}(y \,|\, 1) = \begin{cases} 0.3 & y = 2 \\ 0.3 & y = 1 \\ 0.21 & y = -1 \\ 0.19 & y = -2 \end{cases}$$

Assume that $P_H(-1) = P_H(1) = 1/2$. We consider a hypothesis problem of estimating H after observing Y. Determine whether the following statistics are sufficient statistics or not. Provide a justification for your answer.

a) (2 points) $T_1(Y) = \text{argmax}_h P_{Y|H}(Y \,|\, h)$.

b) (2 points) $T_2(Y) = \Big(T_1(Y), P_{Y|H}\big(Y \,|\, T_1(Y)\big)\Big)$, i.e. a tuple composed of $T_1(Y)$ and the likelihood of observing Y conditioned on $H = T_1(Y)$.

c) (4 points) $T_3(Y) = P_{Y|H}\big(Y \,|\, T_1(Y)\big)/P_{Y|H}\big(Y \,|\, -T_1(Y)\big)$.

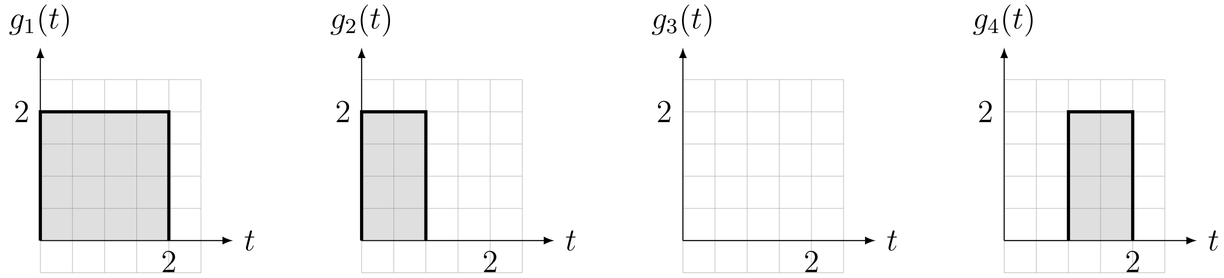
Now, we introduce another hypothesis with conditional probability distribution,

$$P_{Y|H}(y \,|\, 0) = \begin{cases} 0.27 & y = 2 \\ 0.26 & y = 1 \\ 0.24 & y = -1 \\ 0.23 & y = -2 \end{cases}$$

with $P_H(0) = P_H(-1) = P_H(1) = 1/3$.

d) (4 points) Is $T_3(Y)$ a sufficient statistic? Provide a justification for your answer.

Problem 2. Consider the following waveforms,



a) (2 points) Find an orthonormal basis $\psi_1(t)$ and $\psi_2(t)$ for these waveforms such that the support of these two basis waveforms are disjoint.

b) (2 points) Do these waveforms form a minimal energy signal set? If it is not, give a minimal energy set by applying a translation.

We will denote the waveforms corresponding to the minimal energy signal set as $\tilde{g}_i(t)$. Let us assume that the received waveform $R(t)$ is given by $R(t) = \tilde{g}_i(t) + N(t)$, where i is the transmitted message and $N(t)$ is white Gaussian noise of intensity $N_0/2 = 1$. Let $Y_1 = \langle R(t), \psi_1(t) \rangle$ and $Y_2 = \langle R(t), \psi_2(t) \rangle$. Let us assume that all messages are equiprobable.

c) (4 points) Conditioned on i-th message is being sent, what is the distribution of (Y_1, Y_2).

d) (4 points) What is the MAP decision region and the corresponding error probability of this transmission scheme?

Now, we will assume that the noise process $N(t)$ is Gaussian but not white, and, for all t,

$$\mathbb{E}\big[N(t)N(t-\tau)\big] = \begin{cases} 1 - |\tau| & |\tau| < 1 \\ 0 & \text{otherwise.} \end{cases}.$$

The receiver does not know of this new noise model, hence it still computes (Y_1, Y_2) and forms its decision based on the decision region that you have developed in d).

e) (4 points) Is (Y_1, Y_2) a sufficient statistic?

f) (4 points) What is the error probability of the receiver under this new noise model? It is sufficient to give express the error probability as an integral with an explicit integration region.

PROBLEM 3. Consider a binary hypothesis testing problem with the hypothesis $H \in \{0, 1\}$ and the observation $Y \in \mathcal{Y}$. Unlike the standard decision problem where the decoder has to produce and estimate $\hat{H} \in \{0, 1\}$, in this problem decoder can produce an estimate $\hat{H} \in \{0, 1, ?\}$. The symbol ? gives the decoder the option to say "I don't know". Let D_0, D_1, and $D_?$ denote the subsets of observation space $\mathcal{Y}$ for which the decoder declares $0, 1$ and ? respectively.

a) (4 points) We say the decoder makes an error if $\hat{H} \neq ?$ and $\hat{H} \neq H$. Show that the probability of error, P_e, equals

$$P_e = \sum_{y \in D_1} p_H(0) p_{Y|H}(y|0) + \sum_{y \in D_0} p_H(1) p_{Y|H}(y|1).$$

b) (4 points) Show that the probability that $\hat{H} = ?$ equals

$$P_? = \sum_{y \in D_?} p_H(0) p_{Y|H}(y|0) + p_H(1) p_{Y|H}(y|1).$$

c) (4 points) Suppose there is a unit cost for making an error (in the sense of (a)) and a cost of $c_? \geq 0$ for declaring "I don't know". Find the decision rule that minimizes the expected cost (i.e., $P_e + c_? P_?$) and express it in terms of the a posteriori probabilities of the hypotheses.

d) (4 points) Consider now the the m-ary case, with costs as in (c). Show that the rule that that minimizes the expected cost is of the form

$$\hat{H}(y) = \begin{cases} ? & \text{(condition)} \\ \hat{H}_{MAP}(y) & \text{otherwise} \end{cases}$$

and determine the appropriate expression for (condition).

Problem 4. Consider a binary hypothesis testing problem where the observation Y has conditional distribution $p_{Y|H}$. Suppose that we use the MAP rule to decide on $\hat{H}$.

a) (4 points) Show that

$$\Pr(error) = \sum_y \min\{p_H(0)p(y|0),\ p_H(1)p(y|1)\}.$$

For $s \in [0,1]$ define $\mu(s) := \sum_y p_{Y|H}(y|0)^{1-s} p_{Y|H}(y|1)^s$.

b) (4 points) Show that for any $s \in [0,1]$

$$\Pr(error) \leq P_H(0)^{1-s} P_H(1)^s \mu(s).$$

Hint: for non-negative a, b, and any $s \in [0,1], \min\{a,b\} \leq a^{1-s}b^s$.

c) (4 points) Show that $\frac{d\mu(s)}{ds} = \sum_y p_{Y|H}(y|0)^{1-s} p_{Y|H}(y|1)^s \Lambda(y)$
where $\Lambda(y) = \ln[p_{Y|H}(y|1)/p_{Y|H}(y|0)]$, and $\frac{d^2\mu(s)}{ds^2} \geq 0$.

d) (4 points) Show that when $P_H(0) = P_H(1) = 1/2$, and $\mu(s) = \mu(1-s)$ for every $s \in [0,1]$, the upperbound in (b) is minimized when $s = 1/2$.

e) (4 points) Suppose $\mu(s) = \mu(1-s)$ for every $s \in [0,1]$ but $P_H(0) > P_H(1)$. Show that the $s \in [0,1]$ that minimizes the upper bound in (b) satisfies $s > 1/2$.