ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 18 Principles of Digital Communications
Midterm exam Apr. 12, 2019

4 problems, each sub-problem worth 4 points.
165 minutes.
1 sheet (2 pages) of notes allowed.

Good Luck!

Please write your name on each sheet of your answers.

Please write the solution of each problem on a separate sheet.

Problem 1. Consider a binary hypothesis testing problem. When the hypothesis H is i, ($i = 0$ or 1), the observations $Y = (Y_1, \dots, Y_n)$ are given by

$$Y_j = s_i a_j Z_j, \qquad j = 1, \dots, n$$

where $s_0, s_1, a_1, a_2, \dots, a_n$ are known real constants, and $Z_1, \dots, Z_n$ are independent exponentially-distributed random variables with unit mean, i.e., each has density $f_{Z_i}(z) = \exp(-z)\mathbb{1}_{z \geq 0}(z)$. For such a random variable $\mathbb{E}[Z] = \text{Var}[Z] = 1$.
Let T be a statistic computed from observations $Y_1, \dots, Y_n$ as $T = \sum_j b_j Y_j$. Let m_i and v_i be the expectation and variance of T when $H = i$.

(a) Consider $q = (m_0 - m_1)^2/(v_0 + v_1)$ as a measure of the quality of the estimator T. (To have high quality, T should have a large difference in its means compared to its variances). Show that

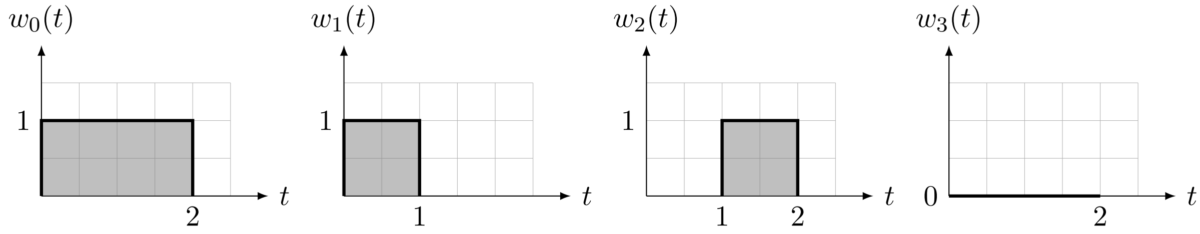
$$q = \frac{(s_0 - s_1)^2}{s_0^2 + s_1^2} \frac{\left(\sum_j a_j b_j\right)^2}{\sum_j (a_j b_j)^2}.$$

(b) Show that $(\sum_j a_j b_j)^2 \leq n \sum_j (a_j b_j)^2$, so that $q \leq n(s_0 - s_1)^2/(s_0^2 + s_1^2)$.

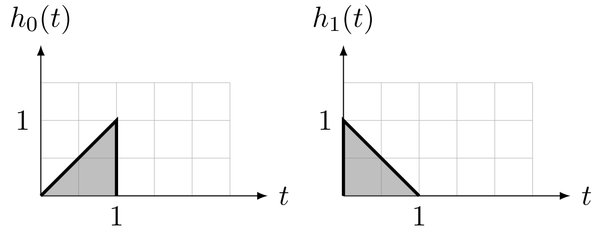
(c) Find a choice of $(b_1, \dots, b_n)$ such that the quality of the statistic is maximized.

(d) Show that for the choice in (c), T is a sufficient statistic.

Problem 2. Consider a signal set with the following four signals.



Each signal is equally likely to be chosen for transmission over the AWGN channel with spectral density $N_0/2$. At the receiver, we are restricted to process the received signal $R(t)$ with the following filters:



We can thus base our decision only on samples obtained at the output of the two filters $Y_{0i} = (R * h_0)(t_{0i})$ and $Y_{1i} = (R * h_1)(t_{1i})$.

(a) Show that we can implement the MAP optimal decision rule inspite of the above restriction by appropriately choosing $\{t_{0i} : i = 1, \ldots, n_0\}$ and $\{t_{1i} : i = 1, \ldots, n_1\}$.

(b) Find a translation of the signal set to minimize the average energy. Sketch the new signal set.

(c) Suppose the signal set in (b) is used for communication with the MAP decoder. What is the error probability?

(d) What is the error probability of the implementation in (a)?

PROBLEM 3. Suppose signals $w_0(t), w_1(t), w_2(t)$ are designed for a communication system such that

$$\langle w_i, w_j \rangle = \begin{cases} 1, & i = j \\ \beta, & i \neq j \end{cases}.$$

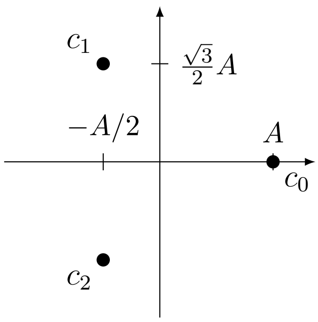
We assume all signals are equiprobable.

(a) Show that $\beta \leq 1$.

(b) Show that $\beta \geq -1/2$.
Hint: Consider $\|w_0 + w_1 + w_2\|^2$.

(c) Translate the signal set to form a minimum average energy constellation with new signals $\tilde{w}_0(t), \tilde{w}_1(t), \tilde{w}_2(t)$. Show that each $\tilde{w}_i(t)$ has energy $E = 2(1-\beta)/3$ and $\langle \tilde{w}_i, \tilde{w}_j \rangle = -E/2$ for $i \neq j$.

(d) Let $e_3(A)$ denote the error probability of the 3-PSK constellation below, where the noise (Z_1, Z_2) has i.i.d. $N(0,1)$ components. (Observe that $\|c_i\|_2 = A$ and c_0, c_1, c_2 are 120° apart.)



Express the error probability of a communication system where the signals $w_0(t), w_1(t), w_2(t)$ are used over an AWGN channel with spectral density $N_0/2$ in terms of β and $e_3(\cdot)$.

PROBLEM 4. Suppose waveforms $w_0(t) = \mathbb{1}_{|t|<\frac{1}{2}}(t)$ and $w_1(t) = -w_0(t)$ are used over the AWGN channel with spectral density $N_0/2$ to transmit 1 bit. As we know, the optimal receiver is a matched filter that forms $Y = (R * h)(t_0)$ with $h(t) = \mathbb{1}_{|t|<\frac{1}{2}}(t)$, $t_0 = 0$, and decides 0 if $Y > 0$ and 1 else. Unfortunately, a faulty clock sets the value of t_0 not to 0 but to 0.5 or -0.5 with equal probability.

(a) What is the error probability of the faulty system?

(b) Suppose that instead of using the matched filter we use a different filter, namely $h(t) = \mathbb{1}_{|t|<1}(t)$. What is the error probability now (assuming the same faulty clock)?

Assume now, that the receiver is aware of the actual sampling time, i.e., whether the sample value Y was taken at $t_0 = -0.5$ or $+0.5$. Let us analyze the error probability of a generic filter $h(t)$. For that purpose, let

$$A_- = \int_{-1}^{0} h(t)\,dt, \qquad E_- = \int_{-\infty}^{0} h(t)^2\,dt,$$
$$A_+ = \int_{0}^{1} h(t)\,dt, \qquad E_+ = \int_{0}^{\infty} h(t)^2\,dt.$$

(c) Show that the error probability can be expressed as

$$P_e = \frac{1}{2}Q\left(\sqrt{\frac{A_+^2}{\sigma^2(E_- + E_+)}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{A_-^2}{\sigma^2(E_- + E_+)}}\right),$$

with $\sigma^2 = \frac{N_0}{2}$.

(d) Show that $A_+^2 \le E_+$, and $A_-^2 \le E_-$.

(e) Use (c), (d) and the fact that $\frac{1}{2}Q(\sqrt{x}) + \frac{1}{2}Q(\sqrt{y}) \ge Q\left(\sqrt{\frac{x+y}{2}}\right)$ to conclude that the filter in (b) has the smallest possible error probability.