ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 36 — Principles of Digital Communications

Final solutions — June 24, 2019

PROBLEM 1. (a)

$$\begin{aligned}
P_e &= \mathbb{P}\left(\hat{H} \neq ?, \hat{H} \neq H\right) \\
&= \mathbb{P}\left(\hat{H} = 1, H = 0\right) + \mathbb{P}\left(\hat{H} = 0, H = 1\right) \\
&= P_H(0)\, \mathbb{P}\left(\hat{H} = 1 \,|\, H = 0\right) + P_H(1)\, \mathbb{P}\left(\hat{H} = 0 \,|\, H = 1\right) \\
&= P_H(0) \int_{\mathcal{R}_1} f_{Y|H}(y|0)\, dy + P_H(1) \int_{\mathcal{R}_0} f_{Y|H}(y|1)\, dy \\
&= \int_{\mathcal{R}_1} b_0(y)\, dy + \int_{\mathcal{R}_0} b_1(y)\, dy.
\end{aligned}$$

$$\begin{aligned}
P_? &= \mathbb{P}\left(\hat{H} = ?\right) \\
&= \mathbb{P}\left(\hat{H} = ?, H = 0\right) + \mathbb{P}\left(\hat{H} = ?, H = 1\right) \\
&= P_H(0)\, \mathbb{P}\left(\hat{H} = ? \,|\, H = 0\right) + P_H(1)\, \mathbb{P}\left(\hat{H} = ? \,|\, H = 1\right) \\
&= P_H(0) \int_{\mathcal{R}_?} f_{Y|H}(y|0)\, dy + P_H(1) \int_{\mathcal{R}_?} f_{Y|H}(y|1)\, dy \\
&= \int_{\mathcal{R}_?} \frac{b_?(y)}{\alpha}\, dy.
\end{aligned}$$

Hence, $P_e + \alpha P_? = \int_{\mathcal{R}_1} b_0(y)\, dy + \int_{\mathcal{R}_0} b_1(y)\, dy + \int_{\mathcal{R}_?} b_?(y)\, dy$.

(b) Let's calculate $P_e + \alpha P_?$ for this decision rule:

$$\begin{aligned}
P_e + \alpha P_? &= \int_{\mathcal{R}_1} b_0(y)\, dy + \int_{\mathcal{R}_0} b_1(y)\, dy + \int_{\mathcal{R}_?} b_?(y)\, dy \\
&= \int_{\{y:\, b_0(y)=b(y)\}} b_0(y)\, dy + \int_{\{y:\, b_1(y)=b(y)\}} b_1(y)\, dy + \int_{\{y:\, b_?(y)=b(y)\}} b_?(y)\, dy \\
&= \int_{\{y:\, b_0(y)=b(y)\}} b(y)\, dy + \int_{\{y:\, b_1(y)=b(y)\}} b(y)\, dy + \int_{\{y:\, b_?(y)=b(y)\}} b(y)\, dy \\
&= \int_{\{y:\, b_0(y)=b(y) \,\vee\, b_1(y)=b(y) \,\vee\, b_?(y)=b(y)\}} b(y)\, dy \\
&= \int_y b(y)\, dy.
\end{aligned}$$

Since this decision rule achieves the lower bound given in the hint, it minimizes $P_e + \alpha P_?$.

(c) Notice that $L = \frac{b_0(y)}{b_1(y)}$. Dividing both sides by $b_1(y)$ for each case of the decision rule $\hat{H}(y)$, we get:

$$\hat{H}(y) = \begin{cases} 1 & L(y) = b'(y), \\ 0 & 1 = b'(y), \\ ? & \alpha(1 + L(y)) = b'(y), \end{cases}$$

where $b'(y) = \min\left(L(y), 1, \alpha(1 + L(y))\right)$. Equivalently,

$$\hat{H}(y) = \begin{cases} 1 & L(y) < 1 \wedge L(y) < \alpha(1 + L(y)), \\ 0 & 1 < L(y) \wedge 1 < \alpha(1 + L(y)), \\ ? & \alpha(1 + L(y)) < L(y) \wedge \alpha(1 + L(y)) < 1. \end{cases}$$

Further simplifying we obtain,

$$\hat{H}(y) = \begin{cases} 1 & L(y) < 1 \wedge L(y) < \frac{\alpha}{1-\alpha}, \\ 0 & 1 < L(y) \wedge \frac{1-\alpha}{\alpha} < L(y), \\ ? & \frac{\alpha}{1-\alpha} \leq L(y) \leq \frac{1-\alpha}{\alpha}, \end{cases}$$

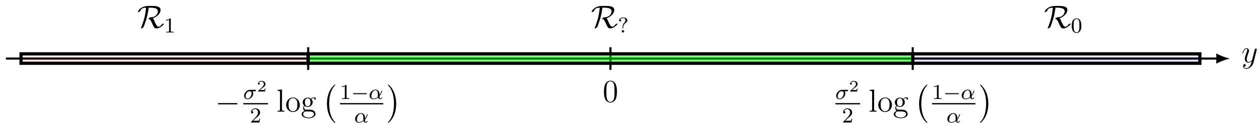
Given the assumption, $L(y) < \frac{\alpha}{1-\alpha}$ implies $L(y) < 1$ and $L(y) > \frac{1-\alpha}{\alpha}$ implies $L(y) > 1$, and these complete the proof.

(d) Let's calculate $L(y)$ for this setup:

$$\begin{aligned} L(y) &= \frac{\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-1)^2}{2\sigma^2}\right)}{\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y+1)^2}{2\sigma^2}\right)} \\ &= \exp\left(\frac{-y^2 + 2y - 1 + y^2 + 2y + 1}{2\sigma^2}\right) \\ &= \exp\left(\frac{2y}{\sigma^2}\right). \end{aligned}$$

So, applying the result of (c), we get:

$$\hat{H}(y) = \begin{cases} 1 & y < -\frac{\sigma^2}{2} \log\left(\frac{1-\alpha}{\alpha}\right), \\ 0 & y > \frac{\sigma^2}{2} \log\left(\frac{1-\alpha}{\alpha}\right), \\ ? & \text{otherwise}. \end{cases}$$

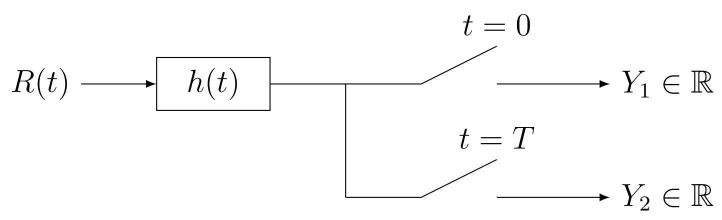


PROBLEM 2. (a) Using Nyquist's criterion, the set $\{\psi(t-kT),\ k\in\mathbb{Z}\}$ forms an orthogonal basis if

$$\sum_k \left|\psi_{\mathcal{F}}(f-\frac{k}{T})\right|^2 = c, \tag{1}$$

for some $c\in\mathbb{R}$. From the figure, we see that (1) holds if $\frac{1}{T}=2$, i.e. $T=\frac{1}{2}$. Moreover $c=1$, hence $\psi(t)$ is *not* a unit-norm pulse but has energy $\mathcal{E}=2$.

(b) Let $R(t)$ denote the noisy channel output. Optimal detection of $\mathcal{S}$ is provided by the ML decoder since all signals are equiprobable. The sufficient statistics $\mathbf{Y}=(Y_1,Y_2)\in\mathbb{R}^2$ are obtained by filtering $R(t)$ by the matched filter of impulse response $h(t)=\psi(-t)/\sqrt{\mathcal{E}}$ and sampling its output at $t=0$ and $t=T$, respectively:



The decision rule is then $\hat{H}=\arg\min_i \|\mathbf{Y}-\mathbf{c}_i\|_2^2$, with

$$\mathbf{c}_0=(0,0),\qquad \mathbf{c}_1=(\sqrt{\mathcal{E}},0),\qquad \mathbf{c}_2=(0,\sqrt{\mathcal{E}}),\qquad \mathbf{c}_3=(\sqrt{\mathcal{E}},\sqrt{\mathcal{E}}).$$

(c) A minimum energy signal set $\tilde{\mathcal{S}}=\{\tilde{w}_0,\tilde{w}_1,\tilde{w}_2,\tilde{w}_3\}$ is obtained by subtracting $m=\mathbb{E}[\mathcal{S}]$ from $\mathcal{S}$:

$$m(t)=\frac{1}{4}\sum_{k=0}^{3}w_k(t)=\frac{1}{2}\psi(t)+\frac{1}{2}\psi(t-T).$$

$$\tilde{w}_0(t)=w_0(t)-m(t)=-\frac{1}{2}\psi(t)-\frac{1}{2}\psi(t-T)=-\frac{\sqrt{\mathcal{E}}}{2}h(t)-\frac{\sqrt{\mathcal{E}}}{2}h(t-T),$$

$$\tilde{w}_1(t)=w_1(t)-m(t)=\frac{1}{2}\psi(t)-\frac{1}{2}\psi(t-T)=\frac{\sqrt{\mathcal{E}}}{2}h(t)-\frac{\sqrt{\mathcal{E}}}{2}h(t-T),$$

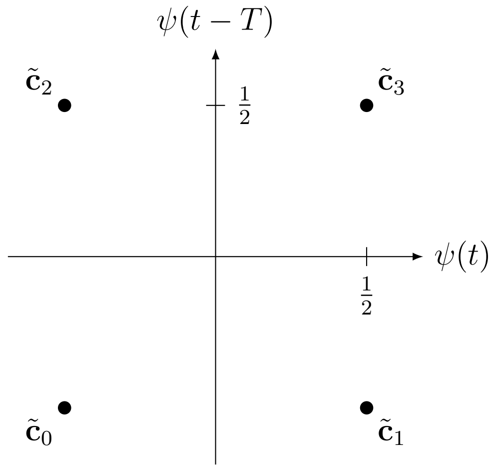
$$\tilde{w}_2(t)=w_2(t)-m(t)=-\frac{1}{2}\psi(t)+\frac{1}{2}\psi(t-T)=-\frac{\sqrt{\mathcal{E}}}{2}h(t)+\frac{\sqrt{\mathcal{E}}}{2}h(t-T),$$

$$\tilde{w}_3(t)=w_3(t)-m(t)=\frac{1}{2}\psi(t)+\frac{1}{2}\psi(t-T)=\frac{\sqrt{\mathcal{E}}}{2}h(t)+\frac{\sqrt{\mathcal{E}}}{2}h(t-T).$$

The codewords $\{\tilde{\mathbf{c}}_0,\tilde{\mathbf{c}}_1,\tilde{\mathbf{c}}_2,\tilde{\mathbf{c}}_3\}$ associated to $\tilde{\mathcal{S}}$ in the $\{\psi(t),\psi(t-T)\}$ basis are

$$\tilde{\mathbf{c}}_0=\left(-\frac{1}{2},-\frac{1}{2}\right),\qquad \tilde{\mathbf{c}}_1=\left(\frac{1}{2},-\frac{1}{2}\right),\qquad \tilde{\mathbf{c}}_2=\left(-\frac{1}{2},\frac{1}{2}\right),\qquad \tilde{\mathbf{c}}_3=\left(\frac{1}{2},\frac{1}{2}\right),$$

and are plotted below:



(d) Simply replace $\{\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ with $\{\hat{\mathbf{c}}_0, \hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3\}$ in (b), where $\hat{\mathbf{c}}_i = \sqrt{\mathcal{E}}\tilde{\mathbf{c}}_i$.

(e) The receiver in (b) corresponds to the ML decoder for a 4-QAM constellation of parameter $d = \mathcal{E}$ and $\sigma^2 = N_0/2$. The error probability of the system is therefore

$$P_e = 2Q\left(\frac{d}{2\sigma}\right) - Q^2\left(\frac{d}{2\sigma}\right) = 2Q\left(\sqrt{\frac{\mathcal{E}}{2N_0}}\right) - Q^2\left(\sqrt{\frac{\mathcal{E}}{2N_0}}\right).$$

(f) The decoding regions of receiver (b) are $\mathcal{R}_i = \left\{\mathbf{Y} \in \mathbb{R}^2 \mid \|\mathbf{Y} - \mathbf{c}_i\|_2^2 \leq \frac{d}{2}\right\}$, which leads to the explicit decoding rule

$$\hat{H}(Y_1, Y_2) = \begin{cases} 0 & Y_1 \leq \frac{d}{2} \cap Y_2 \leq \frac{d}{2}, \\ 1 & Y_1 \geq \frac{d}{2} \cap Y_2 \leq \frac{d}{2}, \\ 2 & Y_1 \leq \frac{d}{2} \cap Y_2 \geq \frac{d}{2}, \\ 3 & Y_1 \geq \frac{d}{2} \cap Y_2 \geq \frac{d}{2}. \end{cases}$$

When using $\tilde{\mathcal{S}}$ with receiver (b), we have $\mathbf{Y}|H = i \sim \mathcal{N}(\hat{\mathbf{c}}_i, \frac{N_0}{2} I_2)$ such that probability of error-free transmission is given by $P_c = \frac{1}{4}\sum_{k=0}^{3} P_c(i)$, where[1]:

[1] In what follows, we replace $N_0/2$ by σ^2 for notational convenience.

$$
\begin{aligned}
P_c(0) &= \Pr\left\{\hat{H} = H \middle| H = 0\right\} = \Pr\left\{Y_1 \le \frac{d}{2},\ Y_2 \le \frac{d}{2} \,\middle|\, H = 0\right\} \\
&= \Pr\left\{[\hat{\mathbf{c}}_0]_1 + \sigma Z_1 \le \frac{d}{2}\right\} \Pr\left\{[\hat{\mathbf{c}}_0]_2 + \sigma Z_2 \le \frac{d}{2}\right\} \\
&= \left[1 - Q\left(\sqrt{\frac{\mathcal{E}}{\sigma^2}}\right)\right]^2, \\
P_c(1) &= \Pr\left\{\hat{H} = H \middle| H = 1\right\} = \Pr\left\{Y_1 \ge \frac{d}{2},\ Y_2 \le \frac{d}{2} \,\middle|\, H = 1\right\} \\
&= \Pr\left\{[\hat{\mathbf{c}}_1]_1 + \sigma Z_1 \ge \frac{d}{2}\right\} \Pr\left\{[\hat{\mathbf{c}}_1]_2 + \sigma Z_2 \le \frac{d}{2}\right\} \\
&= \frac{1}{2}\left[1 - Q\left(\sqrt{\frac{\mathcal{E}}{\sigma^2}}\right)\right], \\
P_c(2) &= P_c(1) \text{ by symmetry}, \\
P_c(3) &= \Pr\left\{\hat{H} = H \middle| H = 3\right\} = \Pr\left\{Y_1 \ge \frac{d}{2},\ Y_2 \ge \frac{d}{2} \,\middle|\, H = 3\right\} \\
&= \Pr\left\{[\hat{\mathbf{c}}_3]_1 + \sigma Z_1 \ge \frac{d}{2}\right\} \Pr\left\{[\hat{\mathbf{c}}_3]_2 + \sigma Z_2 \ge \frac{d}{2}\right\} \\
&= \frac{1}{4}.
\end{aligned}
$$

Therefore the error rate when decoding $\tilde{\mathcal{S}}$ with receiver (b) is

$$
P_e = 1 - P_c = 1 - \frac{1}{16}\left[9 - 12Q\left(\sqrt{\frac{2\mathcal{E}}{N_0}}\right) + 4Q^2\left(\sqrt{\frac{2\mathcal{E}}{N_0}}\right)\right].
$$

PROBLEM 3. (a) The fact that $\psi(t)$ is real-valued implies its Fourier transform is conjugate symmetric. As it also forms an orthogonal collection, the Nyquist criterion must hold:

$$\sum_{k\in\mathbb{Z}} \left|\psi_{\mathcal{F}}\left(f-\frac{k}{T}\right)\right|^2 = C,$$

where $T = 1$ in our case and C is any constant. (We only require the collection to be orthogonal and not orthonormal.) Let us denote slices of the power spectrum as

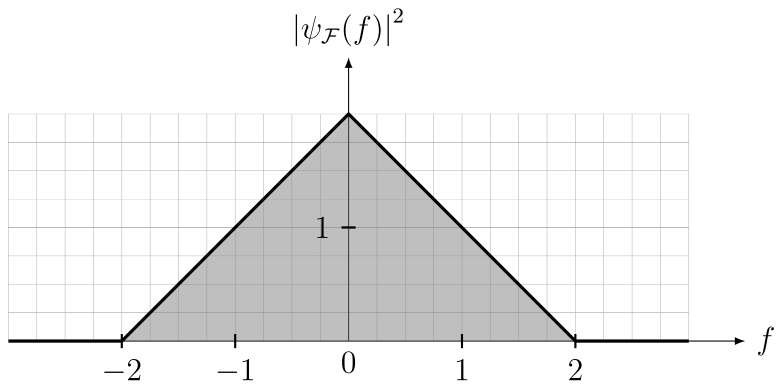
$$\mu_1(f) = |\psi_{\mathcal{F}}(f)|^2\,\mathbb{1}_{0\le f\le \frac{1}{2}} \qquad \mu_2(f) = \left|\psi_{\mathcal{F}}\left(f+\frac{1}{2}\right)\right|^2\mathbb{1}_{0\le f\le \frac{1}{2}}$$

$$\mu_3(f) = |\psi_{\mathcal{F}}(f+1)|^2\,\mathbb{1}_{0\le f\le \frac{1}{2}} \qquad \mu_4(f) = \left|\psi_{\mathcal{F}}\left(f+\frac{3}{2}\right)\right|^2\mathbb{1}_{0\le f\le \frac{1}{2}},$$

where $\mu_2(f)$ and $\mu_4(f)$ are currently unknown. The Nyquist criterion is fulfilled provided that:

$$\mu_1(f) + \mu_3(f) + \mu_2(-f) + \mu_4(-f) = C\mathbb{1}_{0\le f\le \frac{1}{2}}.$$

One possible spectrum that fulfills the criterion is:



(b) The autocorrelation function can be written as:

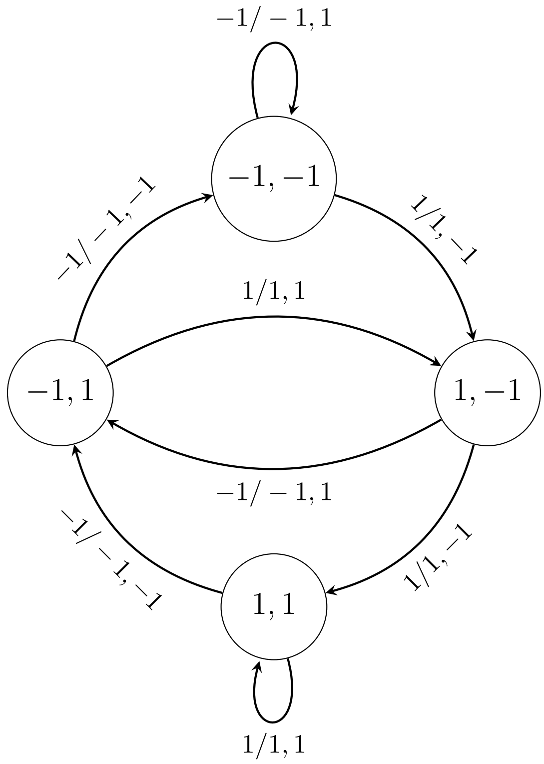
$$E[X_iX_{i+k}] = \begin{cases} 1 & k=0,\\ 0 & \text{otherwise.}\end{cases}$$

The $k = 0$ case is obvious. For the other case $E[X_iX_{i+k}] \neq 0$ can only happen if they share random variables. This is only possible for the following cases:

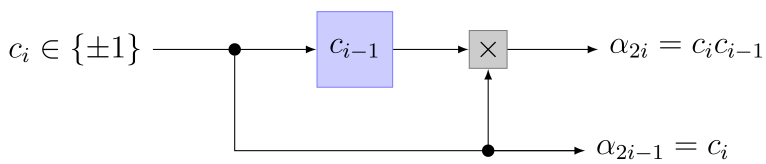
$$\begin{aligned} E[X_{2i}X_{2i-1}] &= E[b_i^2 b_{i-2}] = E[b_i^2]E[b_{i-2}] = 0\\ E[X_{2(i+2)}X_{2i-1}] &= E[b_i^2 b_{i+2}] = E[b_i^2]E[b_{i+2}] = 0\\ E[X_{2(i+2)}X_{2i}] &= E[b_i^2 b_{i+2} b_{i-2}] = E[b_i^2]E[b_{i+2}]E[b_{i-2}] = 0 \end{aligned}$$

As for these cases the correlation is also 0, therefore $E[X_iX_{i+k}] = 0$ if $k \neq 0$.

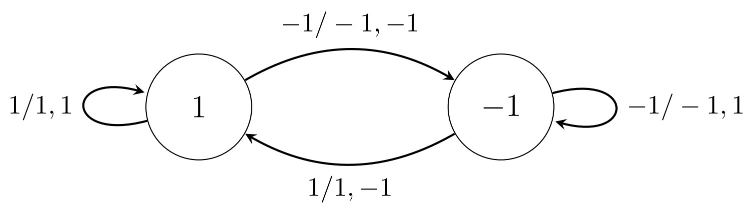
(c) From the problem, we see that the encoder will need order 2 memory due to the b_{i-2} term. Therefore, we will take $b_{i-1}b_{i-2}$ as the state. At each iteration, the encoder will take b_i, then return X_{2i-1} and X_{2i} as its output. The encoder can be expressed as the following state diagram.



(d) The encoder over even-indexed sequences c_i is as follows:



Its state diagram is given below:



The circuit and state diagram corresponding to odd-indexed inputs d_i is identical to above. If the output of the odd-indexd inputs are $\beta_{2i-1}\beta_{2i}$, then the 4-state base encoder is obtained by interleaving outputs of the 2-state encoders as such: $\alpha_1\alpha_2\beta_1\beta_2\alpha_3\alpha_4\beta_3\beta_4\ldots$.

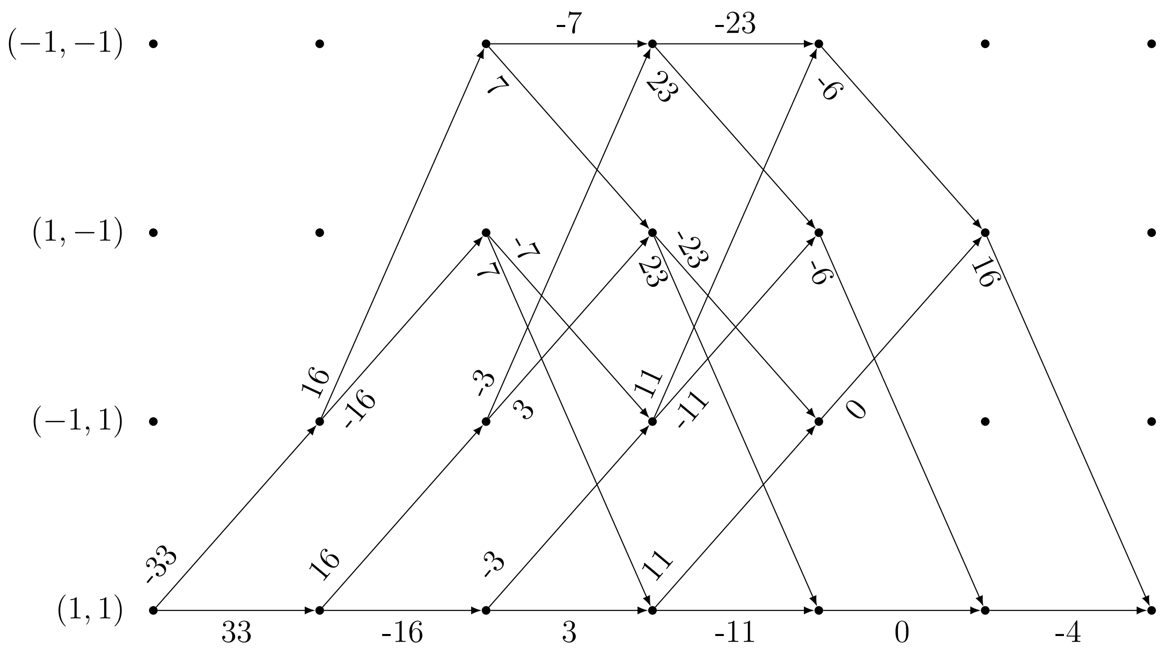
(e) As we are working with a AWGN channel, the channel metric is equal to

$$\log(f(\mathbf{Y}|\mathbf{X})) = -\frac{1}{2\sigma^2}\|\mathbf{Y} - a\mathbf{X}\|_2^2 - \frac{12}{2}\log(2\pi\sigma^2),$$

where $a = \mathcal{E}\int_{-\infty}^{\infty}|\psi_{\mathcal{F}}(f)|^2 df$ as our basis is real-valued and unnormalized. Therefore we want to find a solution to the following optimization problem:

$$\begin{aligned}
\arg\max_{X_1,..,X_{12}} \sum_{i=1}^{12} \log(f(Y_i|X_i)) &= \arg\min_{X_1,..,X_{12}} \sum_{i=1}^{12} (Y_i - aX_i)^2 \\
&= \arg\min_{X_1,..,X_{12}} \sum_{i=1}^{12} Y_i^2 + (aX_i)^2 - 2aY_iX_i \\
&= \arg\max_{X_1,..,X_{12}} \sum_{i=1}^{12} Y_iX_i,
\end{aligned}$$

where the domain of optimization is taken over all possible sequences $X_1, ..., X_{12}$ generated by the encoder in (d). Note that the choice of spectrum at (a) and the channel variance are irrelevant as the final objective function does not depend on a or σ^2. To solve this, consider the following trellis denoting the corresponding objective function $Y_{2i}X_{2i} + Y_{2i-1}X_{2i-1}$ for each path.



From the trellis, the channel metric is maximized if we choose input sequence $\{1, -1, 1, 1\}$ with the path metric total of 72.