ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 35 Principles of Digital Communications
Solutions to Final Exam Jul. 7, 2017

SOLUTION 1.

(a) We have $\text{tri}_{\mathcal{F}}(f) = \text{rect}^2_{\mathcal{F}}(f) = \text{sinc}^2(f)$ because $\text{tri}(t) = (\text{rect} * \text{rect})(t)$. Now since $\psi(t) = \text{sinc}(t) \cdot \text{tri}(t)$, we have

$$\begin{aligned}\psi_{\mathcal{F}}(f) &= (\text{sinc}_{\mathcal{F}} * \text{tri}_{\mathcal{F}})(f) = (\text{rect} * \text{sinc}^2)(f)\\ &= \int_{-\infty}^{+\infty} \text{rect}(f-u)\,\text{sinc}^2(u)du = \int_{f-\frac{1}{2}}^{f+\frac{1}{2}} \text{sinc}^2(u)du = a(f).\end{aligned}$$

On the other hand, we have $\mathbb{E}[X_i] = 0$ and

$$K_X[k] = \mathbb{E}[X_{i+k}X_i^*] = \mathcal{E}\mathbb{1}\{k=0\}.$$

Thus,

$$S_X(f) = |\psi_{\mathcal{F}}(f)|^2 \sum_k K_X[k] e^{-\mathrm{j}2\pi k f} = \mathcal{E}a^2(f).$$

Now since $h(t) = \delta(t) - 2\delta(t-1/4) + \delta(t-1/2)$, we have

$$h_{\mathcal{F}}(f) = 1 - 2e^{-\mathrm{j}\frac{\pi f}{2}} + e^{-\mathrm{j}\pi f} = \left(1 - e^{-\mathrm{j}\frac{\pi f}{2}}\right)^2.$$

Therefore,

$$S_Y(f) = S_X(f) \cdot |h_{\mathcal{F}}(f)|^2 = \mathcal{E}a^2(f) \cdot \left|1 - e^{-\mathrm{j}\frac{\pi f}{2}}\right|^4.$$

Since $a(f) > 0$ for every $f \in \mathbb{R}$, we have $S_Y(f) = 0$ if and only if $1 - e^{-\mathrm{j}\frac{\pi f}{2}} = 0$. Therefore,

$$S_Y(f) = 0 \quad \Leftrightarrow \quad f = 4m \text{ for some } m \in \mathbb{Z}.$$

(b) We have

$$\mathbb{E}[X_i^2] = s^2\left(\mathbb{E}[D_i^2] + 2\alpha\mathbb{E}[D_iD_{i-4}] + \alpha^2\mathbb{E}[D_{i-4}^2]\right) = s^2(1+0+\alpha^2) = s^2(1+\alpha^2).$$

Hence,

$$s = \pm\sqrt{\frac{\mathcal{E}}{1+\alpha^2}}.$$

We still have $\mathbb{E}[X_i] = 0$. However,

$$\begin{aligned}K_X[k] &= \mathbb{E}\left[X_{i+k}X_i^*\right]\\ &= s^2\left(\mathbb{E}[D_{i+k}D_i] + \alpha\mathbb{E}[D_{i+k}D_{i-4}] + \alpha\mathbb{E}[D_{i+k-4}D_i] + \alpha^2\mathbb{E}[D_{i+k-4}D_{i-4}]\right)\\ &= s^2\left((1+\alpha^2)\mathbb{1}\{k=0\} + \alpha\mathbb{1}\{k=-4\} + \alpha\mathbb{1}\{k=4\}\right)\\ &= \mathcal{E}\left(\mathbb{1}\{k=0\} + \frac{\alpha}{1+\alpha^2}\mathbb{1}\{k=-4\} + \frac{\alpha}{1+\alpha^2}\mathbb{1}\{k=4\}\right).\end{aligned}$$

Therefore,

$$\begin{aligned} S_X(f) &= |\psi_{\mathcal{F}}(f)|^2 \sum_k K_X[k] e^{-\mathrm{j}2\pi k f} \\ &= \mathcal{E} \cdot a^2(f) \left(1 + \frac{\alpha}{1+\alpha^2} e^{\mathrm{j}8\pi f} + \frac{\alpha}{1+\alpha^2} e^{-\mathrm{j}8\pi f}\right) \\ &= \mathcal{E} \cdot a^2(f) \left(1 + \frac{2\alpha}{1+\alpha^2} \cos(8\pi f)\right), \end{aligned}$$

and

$$S_Y(f) = S_X(f) \cdot |h_{\mathcal{F}}(f)|^2 = \mathcal{E} a^2(f) \left(1 + \frac{2\alpha}{1+\alpha^2} \cos(8\pi f)\right) \cdot \left|1 - e^{-\mathrm{j}\frac{\pi f}{2}}\right|^4.$$

We have several cases:

- If $|\alpha| \neq 1$, we have $\left|\frac{2\alpha}{1+\alpha^2}\right| < 1$ and so $1 + \frac{2\alpha}{1+\alpha^2}\cos(8\pi f) \neq 0$ for every $f \in \mathbb{R}$. Hence,

$$\begin{aligned} S_Y(f) = 0 \quad &\Leftrightarrow \quad 1 - e^{-\mathrm{j}\frac{\pi f}{2}} = 0 \\ &\Leftrightarrow \quad f = 4m \text{ for some } m \in \mathbb{Z}. \end{aligned}$$

- If $\alpha = -1$, we have

$$S_Y(f) = S_X(f) \cdot |h_{\mathcal{F}}(f)|^2 = \mathcal{E} a^2(f) \left(1 - \cos(8\pi f)\right) \cdot \left|1 - e^{-\mathrm{j}\frac{\pi f}{2}}\right|^4.$$

  Hence,

$$\begin{aligned} S_Y(f) = 0 \quad &\Leftrightarrow \quad 1 - e^{-\mathrm{j}\frac{\pi f}{2}} = 0 \text{ or } 1 - \cos(8\pi f) = 0 \\ &\Leftrightarrow \quad f = 4m \text{ or } f = \frac{m}{4} \text{ for some } m \in \mathbb{Z} \\ &\Leftrightarrow \quad f = \frac{m}{4} \text{ for some } m \in \mathbb{Z}. \end{aligned}$$

- If $\alpha = 1$, we have

$$S_Y(f) = S_X(f) \cdot |h_{\mathcal{F}}(f)|^2 = \mathcal{E} a^2(f) \left(1 + \cos(8\pi f)\right) \cdot \left|1 - e^{-\mathrm{j}\frac{\pi f}{2}}\right|^4.$$

  Hence,

$$\begin{aligned} S_Y(f) = 0 \quad &\Leftrightarrow \quad 1 - e^{-\mathrm{j}\frac{\pi f}{2}} = 0 \text{ or } 1 + \cos(8\pi f) = 0 \\ &\Leftrightarrow \quad f = 4m \text{ or } f = \frac{2m+1}{8} \text{ for some } m \in \mathbb{Z}. \end{aligned}$$

(c) Since $D_i^2 = D_{i-1}^2 = 1$, we have $X_i = s(D_i + D_{i-1})$. Hence

$$\mathbb{E}[X_i^2] = s^2 \left(\mathbb{E}[D_i^2] + 2\mathbb{E}[D_i D_{i-1}] + \mathbb{E}[D_{i-1}^2]\right) = s^2(1+0+1) = 2s^2.$$

and

$$s = \pm\sqrt{\frac{\mathcal{E}}{2}}.$$

We still have $\mathbb{E}[X_i] = 0$. However,

$$\begin{aligned} K_X[k] &= \mathbb{E}\left[X_{i+k}X_i^*\right] \\ &= s^2\left(\mathbb{E}[D_{i+k}D_i] + \mathbb{E}[D_{i+k}D_{i-1}] + \mathbb{E}[D_{i+k-1}D_i] + \mathbb{E}[D_{i+k-1}D_{i-1}]\right) \\ &= s^2\left(2\mathbb{1}\{k=0\} + \mathbb{1}\{k=-1\} + \mathbb{1}\{k=1\}\right) \\ &= \mathcal{E}\left(\mathbb{1}\{k=0\} + \frac{1}{2}\mathbb{1}\{k=-1\} + \frac{1}{2}\mathbb{1}\{k=1\}\right). \end{aligned}$$

Hence,

$$\begin{aligned} S_X(f) &= |\psi_{\mathcal{F}}(f)|^2 \sum_k K_X[k]e^{-\mathrm{j}2\pi kf} \\ &= \mathcal{E}\cdot a^2(f)\left(1 + \frac{1}{2}e^{\mathrm{j}2\pi f} + \frac{1}{2}e^{-\mathrm{j}2\pi f}\right) \\ &= \mathcal{E}\cdot a^2(f)\left(1+\cos(2\pi f)\right), \end{aligned}$$

and

$$S_Y(f) = S_X(f)\cdot|h_{\mathcal{F}}(f)|^2 = \mathcal{E}a^2(f)\left(1+\cos(2\pi f)\right)\cdot\left|1-e^{-\mathrm{j}\frac{\pi f}{2}}\right|^4.$$

Therefore,

$$\begin{aligned} S_Y(f) = 0 \quad &\Leftrightarrow \quad 1 - e^{-\mathrm{j}\frac{\pi f}{2}} = 0 \;\text{ or }\; 1+\cos(2\pi f) = 0 \\ &\Leftrightarrow \quad f = 4m \;\text{ or }\; f = \frac{2m+1}{2} \;\text{ for some }\; m\in\mathbb{Z}. \end{aligned}$$

(d) Since $1 \notin \{4m : m\in\mathbb{Z}\}$ and $1 \notin \left\{\frac{2m+1}{2} : m\in\mathbb{Z}\right\}$, we have $S_Y(1) \neq 0$ for the method in (c).

On the other hand, since $1 \notin \{4m : m\in\mathbb{Z}\}$, $1 \notin \left\{\frac{2m+1}{8} : m\in\mathbb{Z}\right\}$ and $1 \in \left\{\frac{m}{4} : m\in\mathbb{Z}\right\}$, the only value of α in (b) for which we have $S_Y(1) = 0$ is $\alpha = -1$.

Solution 2.

(a) Since $w_E(t) = \sum_j X_j \operatorname{sinc}(t-j)$, we have $w_{E,\mathcal{F}}(f) = \sum_j X_j \operatorname{rect}(f)e^{-\mathrm{j}2\pi jf}$. Hence $w_{E,\mathcal{F}}(f) = 0$ for $f \notin \left(-\frac{1}{2},\frac{1}{2}\right)$.

Now since $w_{\mathcal{F}}(f) = \frac{1}{\sqrt{2}}w_{E,\mathcal{F}}(f-f_c)$ for $f>0$, the equality $w_{\mathcal{F}}(f_c-f) = w_{\mathcal{F}}(f_c+f)^*$ implies that $w_{E,\mathcal{F}}(-f) = w_{E,\mathcal{F}}(f)^*$ for $f\in\left(-\frac{1}{2},\frac{1}{2}\right)$. But $w_{E,\mathcal{F}}(f) = 0$ for $f\notin\left(-\frac{1}{2},\frac{1}{2}\right)$. Therefore, $w_{E,\mathcal{F}}(-f) = w_{E,\mathcal{F}}(f)^*$ for every $f\in\mathbb{R}$, which implies that $w_E(t)$ is a real signal, i.e., $X_j\in\{+1,-1\}$ for all $j\in\mathbb{Z}$.

(b) Since $w_{\mathcal{F}}(f) = \frac{1}{\sqrt{2}}w_{E,\mathcal{F}}(f-f_c)$ for $f>0$, the equality $w_{\mathcal{F}}(f_c-f) = -w_{\mathcal{F}}(f_c+f)^*$ implies that $w_{E,\mathcal{F}}(-f) = -w_{E,\mathcal{F}}(f)^*$ for $f\in\left(-\frac{1}{2},\frac{1}{2}\right)$. But $w_{E,\mathcal{F}}(f) = 0$ for $f\notin\left(-\frac{1}{2},\frac{1}{2}\right)$. Therefore, $w_{E,\mathcal{F}}(-f) = -w_{E,\mathcal{F}}(f)^*$ for every $f\in\mathbb{R}$, which implies that $w_E(t)$ is a pure-imaginary signal, i.e., $X_j\in\{+\mathrm{j},-\mathrm{j}\}$ for all $j\in\mathbb{Z}$.

(c) If $\theta = 0$,

$$\begin{aligned} \Re\{R_E(t)\} &= A\cdot\Re\{w_E(t)\} + N_R(t), \\ \Im\{R_E(t)\} &= A\cdot\Im\{w_E(t)\} + N_I(t), \end{aligned}$$

where $N_R(t)$ and $N_I(t)$ are independent white Gaussian noise processes of power spectral density $\frac{N_0}{2}$.

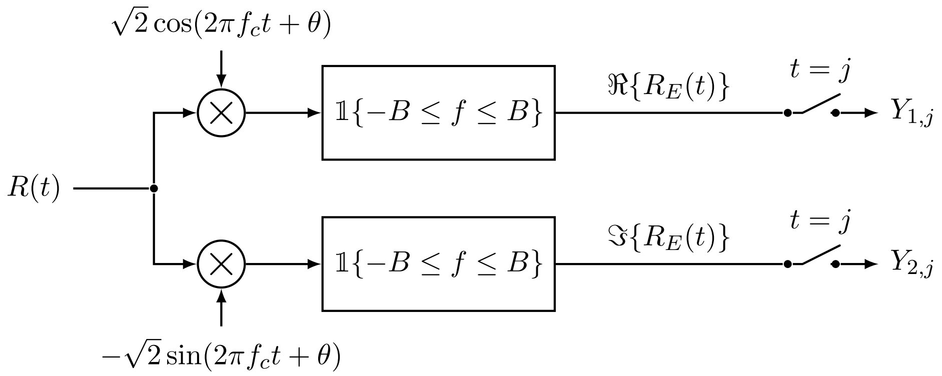
A sufficient statistic to estimate X_j from the received signal is obtained by computing the (complex-valued) inner products

$$Y_j = \langle R_E(t), \mathrm{sinc}(t-j) \rangle,$$

or equivalently, pairs of real-valued inner products

$$Y_{1,j} = \langle \Re\{R_E(t)\}, \mathrm{sinc}(t-j) \rangle \quad \text{and} \quad Y_{2,j} = \langle \Im\{R_E(t)\}, \mathrm{sinc}(t-j) \rangle.$$

To this end, one in principle has to filter the outputs of the down-converter using matched filters of impulse response $\mathrm{sinc}^*(-t)$ and sample the outputs of the filters at times $t = j, j \in \mathbb{Z}$. However, a filter with impulse response $\mathrm{sinc}^*(-t)$ is nothing but a low-pass filter with frequency response $\mathbb{1}\{-\frac{1}{2} \le f \le \frac{1}{2}\}$ which is already included in the down-converter. Thus, it is sufficient to sample the output of the down-converters directly to obtain the desired sufficient statistics.

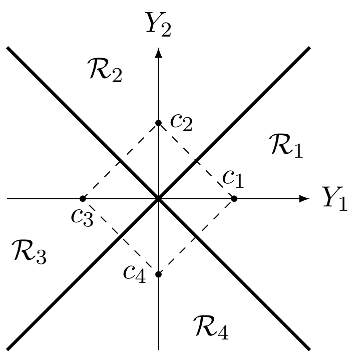


(d) We have the following hypothesis testing problem:

$$\text{under } H = i: \qquad Y = c_i + Z,$$

where $Z \sim \mathcal{N}(0, \frac{N_0}{2} I_2)$ and $c_1 = [A, 0]$, $c_2 = [0, A]$, $c_3 = [-A, 0]$, and $c_4 = [0, -A]$.

For an AWGN setting, the ML decision rule will be the minimum distance decision rule with the following decision regions:



This is a 4-PSK constellation and the probability of error of an ML decoder for such a constellation is

$$P_e = 2Q\left(\frac{A}{\sqrt{N_0}}\right) - Q\left(\frac{A}{\sqrt{N_0}}\right)^2.$$

(e) Since the decision regions in (d) do not depend on A, we do not change anything in the decoder of (d), even if A were unknown to the receiver. The average probability of error is given by

$$\begin{aligned}
P_e &= \mathbb{E}\left[2Q\left(\frac{A}{\sqrt{N_0}}\right) - Q\left(\frac{A}{\sqrt{N_0}}\right)^2\right] \\
&= \frac{1}{2}\left(2Q\left(\frac{1}{\sqrt{2N_0}}\right) - Q\left(\frac{1}{\sqrt{2N_0}}\right)^2\right) + \frac{1}{2}\left(2Q\left(\sqrt{\frac{3}{2N_0}}\right) - Q\left(\sqrt{\frac{3}{2N_0}}\right)^2\right) \\
&= Q\left(\frac{1}{\sqrt{2N_0}}\right) - \frac{1}{2}Q\left(\frac{1}{\sqrt{2N_0}}\right)^2 + Q\left(\sqrt{\frac{3}{2N_0}}\right) - \frac{1}{2}Q\left(\sqrt{\frac{3}{2N_0}}\right)^2.
\end{aligned}$$

(f) Using the trigonometric identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ we can see that the output of the top modulator, in presence of the phase difference, is

$$R(t)\cos(\theta) \times \sqrt{2}\cos(2\pi f_c t) - R(t)\sin(\theta) \times \sqrt{2}\sin(2\pi f_c t).$$

Thus, as the low-pass filter is a linear system, the output of the top low-pass filter is:

$$\Re\{R_E(t)\} = \Re\{w_E(t)\}\cos(\theta) + \Im\{w_E(t)\}\sin(\theta) + \cos(\theta)N_R(t) + \sin(\theta)N_I(t).$$

Similarly, we can show that the output of the bottom low-pass filter is:

$$\Im\{R_E(t)\} = \Im\{w_E(t)\}\cos(\theta) - \Re\{w_E(t)\}\sin(\theta) + \cos(\theta)N_I(t) - \sin(\theta)N_R(t).$$

Therefore, the observable $Y = [Y_1, Y_2]$ (under $H = i$) is now equal to

$$Y = R_\theta c_i + R_\theta Z$$

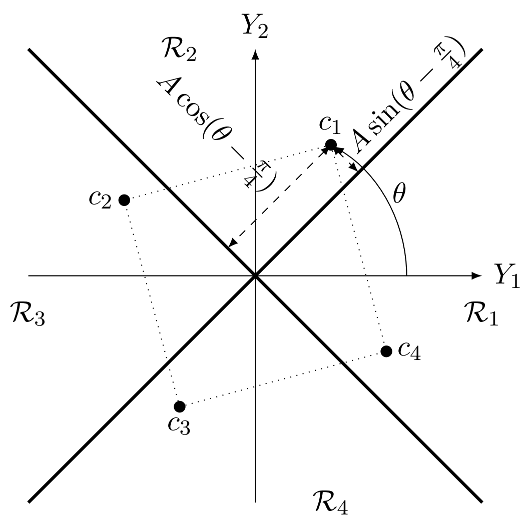
where

$$R_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

is the rotation matrix, the codewords c_i are as in part (d), and $Z \sim \mathcal{N}(0, \frac{N_0}{2}I_2)$. Moreover, we know that $V = R_\theta Z$ has the same statistics as Z. Thus, we can write the observable Y as

$$\text{under } H = i: \qquad Y = R_\theta c_i + V$$

with $V \sim \mathcal{N}(0, \frac{N_0}{2}I_2)$.



Using the above diagram, we can see that the probability of correct guess of the receiver, conditioned on a particular value of A, is

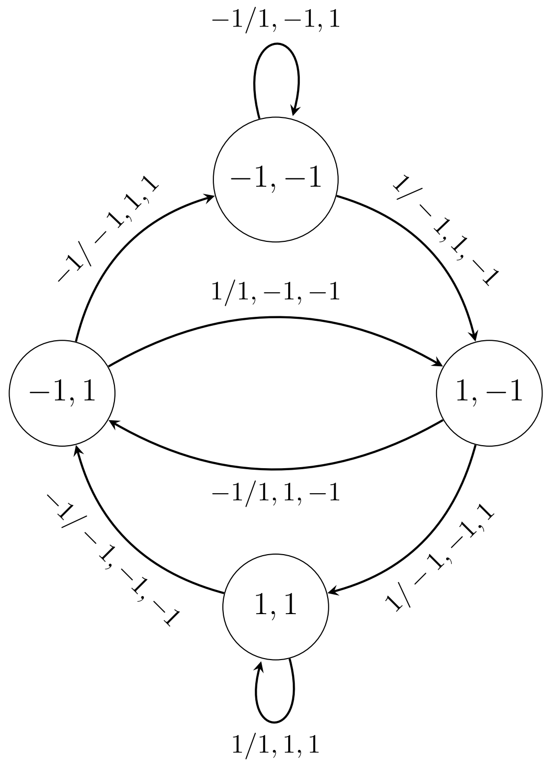
$$\begin{aligned}
P_c &= Q\left(\frac{A\sin(\theta-\frac{\pi}{4})}{\sqrt{N_0/2}}\right) Q\left(-\frac{A\cos(\theta-\frac{\pi}{4})}{\sqrt{N_0/2}}\right) \\
&= \left(1-Q\left(\frac{A\sin(\frac{\pi}{4}-\theta)}{\sqrt{N_0/2}}\right)\right)\left(1-Q\left(\frac{A\cos(\frac{\pi}{4}-\theta)}{\sqrt{N_0/2}}\right)\right) \\
&= 1-Q\left(\frac{A\sin(\frac{\pi}{4}-\theta)}{\sqrt{N_0/2}}\right)-Q\left(\frac{A\cos(\frac{\pi}{4}-\theta)}{\sqrt{N_0/2}}\right)+Q\left(\frac{A\sin(\frac{\pi}{4}-\theta)}{\sqrt{N_0/2}}\right)Q\left(\frac{A\cos(\frac{\pi}{4}-\theta)}{\sqrt{N_0/2}}\right).
\end{aligned}$$

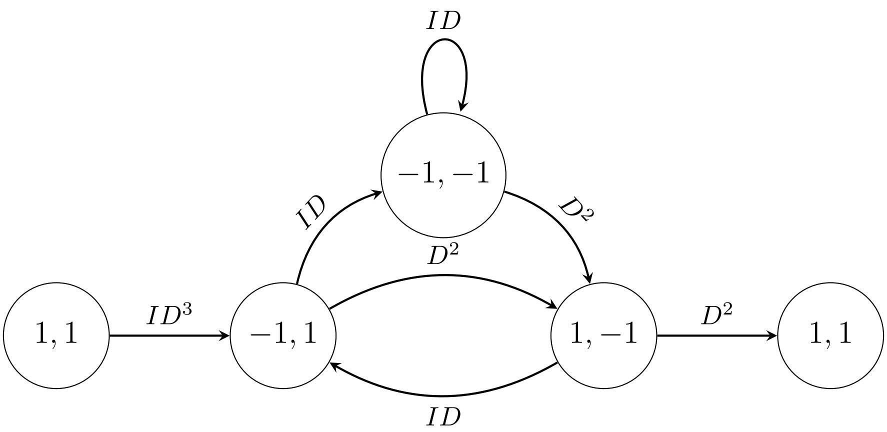
Therefore, the average probability of error is given by

$$\begin{aligned}
P_e &= \mathbb{E}\left[Q\left(\frac{A\sin(\frac{\pi}{4}-\theta)}{\sqrt{N_0/2}}\right)+Q\left(\frac{A\cos(\frac{\pi}{4}-\theta)}{\sqrt{N_0/2}}\right)-Q\left(\frac{A\sin(\frac{\pi}{4}-\theta)}{\sqrt{N_0/2}}\right)Q\left(\frac{A\cos(\frac{\pi}{4}-\theta)}{\sqrt{N_0/2}}\right)\right] \\
&= \frac{1}{2}\left(Q\left(\frac{\sin(\frac{\pi}{4}-\theta)}{\sqrt{N_0}}\right)+Q\left(\frac{\cos(\frac{\pi}{4}-\theta)}{\sqrt{N_0}}\right)-Q\left(\frac{\sin(\frac{\pi}{4}-\theta)}{\sqrt{N_0}}\right)Q\left(\frac{\cos(\frac{\pi}{4}-\theta)}{\sqrt{N_0}}\right)\right) \\
&\quad +\frac{1}{2}\left(Q\left(\frac{\sqrt{3}\sin(\frac{\pi}{4}-\theta)}{\sqrt{N_0}}\right)+Q\left(\frac{\sqrt{3}\cos(\frac{\pi}{4}-\theta)}{\sqrt{N_0}}\right)-Q\left(\frac{\sqrt{3}\sin(\frac{\pi}{4}-\theta)}{\sqrt{N_0}}\right)Q\left(\frac{\sqrt{3}\cos(\frac{\pi}{4}-\theta)}{\sqrt{N_0}}\right)\right).
\end{aligned}$$

Solution 3.

(a) The state diagram and detour flow graph are shown here. The states are labeled as (b_{j-1}, b_{j-2}) and the transitions with $b_j/x_{3j}, x_{3j+1}, x_{3j+2}$.



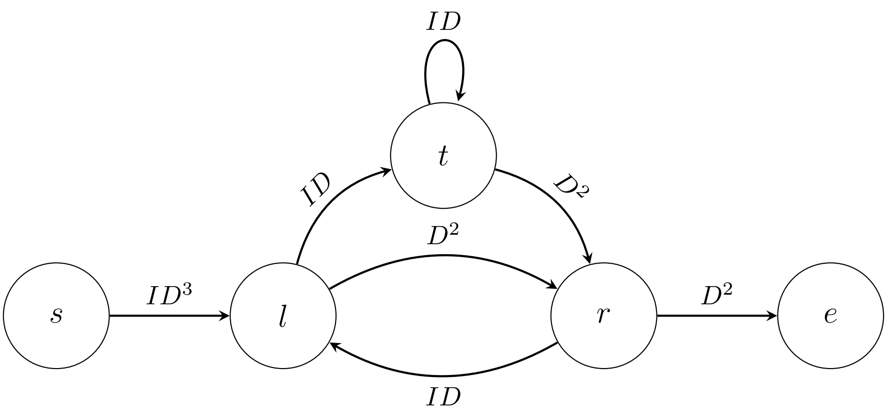


(b) The output to $(1,-1,-1,1,1)$ is $(1,1,1,-1,-1,-1,-1,1,1,-1,1,-1,-1,-1,1)$.

(c) The Bhattacharyya bound is given by

$$
\begin{aligned}
z &= \sum_y \sqrt{\mathbb{P}_{Y|X}(y|1)\mathbb{P}_{Y|X}(y|-1)} \\
&= \sqrt{\mathbb{P}_{Y|X}(1|1)\mathbb{P}_{Y|X}(1|-1)} + \sqrt{\mathbb{P}_{Y|X}(?|1)\mathbb{P}_{Y|X}(?|-1)} + \sqrt{\mathbb{P}_{Y|X}(-1|1)\mathbb{P}_{Y|X}(-1|-1)} \\
&= \sqrt{(1-\epsilon)\frac{\epsilon}{2}} + \sqrt{\left(\frac{\epsilon}{2}\right)^2} + \sqrt{\frac{\epsilon}{2}(1-\epsilon)} \\
&= \sqrt{2\epsilon(1-\epsilon)} + \frac{\epsilon}{2}.
\end{aligned}
$$

(d) Let us relabel the states in the detour flow graph:



We have:

$$
\begin{aligned}
T_l &= ID^3T_s + IDT_r, \\
T_r &= D^2T_l + D^2T_t, \\
T_t &= IDT_l + IDT_t,
\end{aligned}
$$

and

$$T_e = D^2T_r.$$

We have

$$T_l + T_t = ID^3T_s + IDT_r + T_t = ID^3T_s + (ID^3 + ID)(T_l + T_t),$$

hence,

$$T_l + T_t = \frac{ID^3}{1 - ID - ID^3} T_s.$$

Therefore,

$$T_e = D^2 T_r = D^4 (T_l + T_r) = \frac{ID^7}{1 - ID - ID^3}.$$

The generating function of $a(i, d)$ is

$$T(I, D) = \frac{T_e}{T_s} = \frac{ID^7}{1 - ID - ID^3}.$$

We have

$$\frac{\partial T}{\partial I}(I, D) = \frac{D^7(1 - ID - ID^3) - (-D - D^3)ID^7}{(1 - ID - ID^3)^2} = \frac{D^7}{(1 - ID - ID^3)^2}.$$

We conclude that the bit error probability is upper bounded as follows:

$$P_e \leq \frac{\partial T}{\partial I}(1, z) = \frac{z^7}{(1 - z - z^3)^2}.$$

(e) Given the observation $y = (y_0, \ldots, y_{3n-1})$, the ML codeword is given by

$$\arg\max_{x \in \mathcal{C}} p(y|x),$$

where $\mathcal{C}$ represents the set of codewords (i.e., the set of all possible paths on the trellis). Alternately, the ML codeword is given by

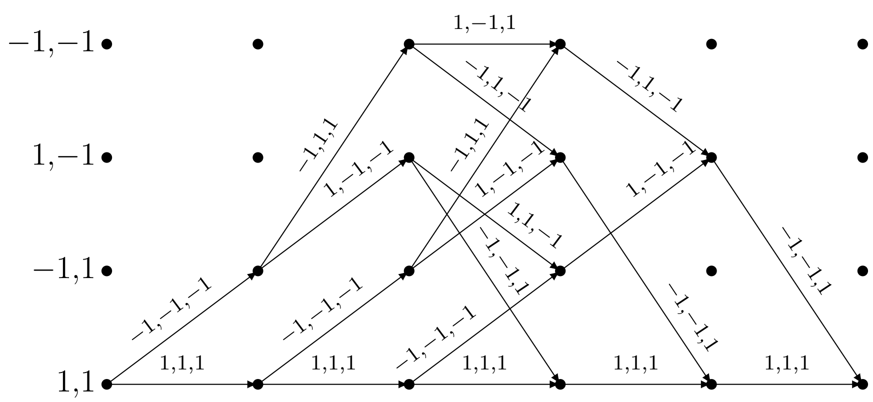
$$\arg\max_{x \in \mathcal{C}} \sum_{i=0}^{3n-1} \log p(y_i|x_i).$$

Hence, a branch metric for the Viterbi decoder is

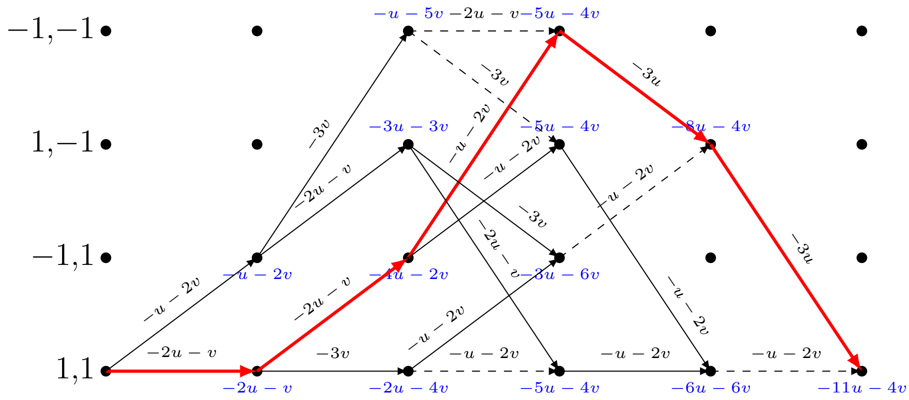
$$\log p(y_i|x_i) = \begin{cases} \log(1 - \epsilon) & \text{if } y_i = x_i, \\ \log\left(\frac{\epsilon}{2}\right) & \text{if } y_i \neq x_i. \end{cases}$$

The decoder chooses the path with the largest metric.

(f) The trellis representing the encoder is shown below:



We display the diagram labeled with edge-metric according to the received sequence and state-metric of the survivor path. We also indicate the survivor paths and the decoding path.



From the figure we can read the decoded sequence $1, -1, -1, 1, 1$.