ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 31 | Principles of Digital Communications
Final exam | July 3, 2023

4 problems, 43 points, 180 minutes
2 sheets (4 pages) of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

PROBLEM 1. In a hypothesis testing problem with hypothesis $H \in \{0, 1, 2\}$, observation $Y \in \mathcal{Y}$, decision $\hat{H} \in \{0, 1, 2\}$, the penalty for deciding j when the true hypothesis is i is $\text{pen}(i, j) = |i - j|$.

(a) (3 pts) Suppose D_0, D_1 and D_2 are the decision regions for a decision rule $\hat{H} : \mathcal{Y} \to \{0, 1, 2\}$. Find $b_0(y)$, $b_1(y)$ and $b_2(y)$ — expressed in terms of $p_{Y|H}(y|i)$ and $p_H(i)$ — so that the expected penalty of the rule is given by

$$\mathbb{E}[\text{pen}(H, \hat{H}(Y))] = \sum_{y \in D_0} b_0(y) + \sum_{y \in D_1} b_1(y) + \sum_{y \in D_2} b_2(y).$$

(b) (2 pts) Show that for any decision rule, $\mathbb{E}[\text{pen}(H, \hat{H})] \geq \sum_{y \in \mathcal{Y}} b(y)$, where

$$\begin{aligned} b(y) = \min\{ & p_{Y|H}(y|1)p_H(1) + 2p_{Y|H}(y|2)p_H(2), \\ & p_{Y|H}(y|0)p_H(0) + p_{Y|H}(y|2)p_H(2), \\ & 2p_{Y|H}(y|0)p_H(0) + p_{Y|H}(y|1)p_H(1) \}. \end{aligned}$$

(c) (2 pts) What is the decision rule that minimizes the expected penalty (in terms of $b(y)$ and/or $b_i(y)$)?

(d) (3 pts) Suppose all hypotheses are equally likely, $\mathcal{Y} = \{0, 1, 2\}$, and

$$p_{Y|H}(y|i) = \begin{cases} 0.4 & y = i, \\ 0.3 & \text{else.} \end{cases}$$

Then, what is the decision rule $\hat{H}(y)$ as an explicit function from $\mathcal{Y}$ to $\{0, 1, 2\}$ that minimizes the expected penalty? What is the MAP rule for this case?

Problem 2. Let $w_i(t)$, $i = 1, \dots, m$ be the waveforms of a communication system designed for an AWGN channel with noise power spectral density $N_0/2$, suppose that $w_i(t) = 0$ whenever $t \notin [0, T]$. Let $\pi(N_0)$ denote the error probability of this system (with its optimal receiver).

Consider now two new systems:

1. The first has waveforms $\tilde{w}_i(t) = \alpha w_i(t)$, $i = 1, \dots, m$ for a (real) scalar $\alpha \neq 0$.
2. The second has waveforms $w_i'(t) = \sum_{j=0}^{r-1} w_i(t - jT)$, $i = 1, \dots, m$; i.e., w_i repeated r times, once every T units of time. (Here r is a positive integer.)

(a) (2 pts) How can we re-use the optimal receiver for the original system to design an optimal receiver for system 1?

Hint: Think of some pre-processing of the received signal $\tilde{R}(t)$ of system 1 before giving it as input to the optimal receiver of the original system.

(b) (2 pts) Express the error probability $\tilde{\pi}(N_0)$ of system 1 (with its optimal receiver), in terms of $\pi(N_0)$.

(c) (3 pts) How can we re-use the optimal receiver for the original system to design an optimal receiver for system 2?

Hint: Think of some pre-processing of the received signal $R'(t)$ of system 2 before giving it as input to the optimal receiver of the original system.

(d) (2 pts) Express the error probability $\pi'(N_0)$ of system 2 (with its optimal receiver), in terms of $\pi(N_0)$.

Problem 3. A bandpass transmitter for four equally likely messages is designed for an AWGN channel as follows:

- The waveform $\psi(t) = \operatorname{sinc}(t)$ is chosen as the Nyquist pulse, upon observing that $\psi(t), \psi(t-1), \psi(t-2), \dots$ form an orthonormal collection. (Note also that the Fourier transform of ψ is $\operatorname{rect}(f) = \mathbb{1}\{|f| < \frac{1}{2}\}$).
- Four codewords $c_1 = (1,0), c_2 = (0,1), c_3 = -c_1, c_4 = -c_2$ are chosen as vectors in $\mathbb{R}^2$. (Note: they are *real*, not complex.)
- At the transmitter, the message $i \in \{1,2,3,4\}$ is first mapped to c_i, then to the baseband waveform $w_{i,E}(t) = \sum_{j=1}^{2} c_{ij}\psi(t-j)$, and finally to the transmitted waveform as

$$\begin{aligned} w_i(t) &= \sqrt{2}\Re\{w_{i,E}(t)\exp(\mathrm{j}2\pi f_c t)\} \\ &= \sqrt{2}w_{i,E}(t)\cos(2\pi f_c t) \qquad \text{with } f_c > \tfrac{1}{2}. \end{aligned}$$

At the receiver, the received signal $R(t)$ is multiplied by $\sqrt{2}\cos(2\pi f_c t)$, to form $R_E(t)$. R_E is passed through a filter with impulse response $\operatorname{sinc}(t)$, and the output of the filter is sampled at times $t_1 = 1$ and $t_2 = 2$. With Y_1 and Y_2 denoting the samples respectively, the vector $Y = (Y_1, Y_2)$ is formed. The i for which c_i is closest to Y (in the Euclidean norm) is the receiver's guess of the transmitted message.

(a) (3 pts) Is the receiver described in the above paragraph optimal? (Note: the procedure in the book would have formed the *complex* waveform $R(t)\sqrt{2}\exp(-\mathrm{j}2\pi f_c t)$ instead of the above R_E. If you claim optimality, you should explain why R_E above leads to the same decision.)

(b) (2 pts) What is the probability of error (in terms of N_0)?

Due to an inaccuracy in circuit design, the frequency of the cosine at the receiver is not f_c but f_c' instead, i.e., $R(t)$ is multiplied by $\sqrt{2}\cos(2\pi f_c' t)$ to form $R_E(t)$. The rest of the receiver is unchanged.

(c) (3 pts) For $x(t) = \operatorname{sinc}(t)\cos(2\pi f_0 t)$ and $y(t) = \operatorname{sinc}(t - t_0)$, show that their inner product satisfies

$$\langle x, y\rangle = \begin{cases} 0 & |f_0| \geq 1, \\ \frac{1}{2}[\operatorname{sinc}(t_0) + (1 - 2|f_0|)\operatorname{sinc}((2|f_0| - 1)t_0)] & \text{else.} \end{cases}$$

Hint: Use Parseval's relationship.

(d) (2 pts) Suppose $|f_c' - f_c| > 1$. What is the error probability?
Hint: Use (c) to show that Y is independent of the transmitted message.

(e) (2 pts) Suppose $f_c' = f_c + \frac{1}{2}$. What is the error probability?

PROBLEM 4. Consider a 3-state encoding device (with states $0, 1, 2$) that accepts a sequence of data bits $b_1, b_2, \ldots$, with $b_i \in \{+1, -1\}$, and produces encoded bits $x_1, x_2, \ldots$ as follows:

Current state	Input bit b_i	Next state	Output x_{2i-1}, x_{2i}
0	$+1$	0	$+1, +1$
0	-1	1	$-1, -1$
1	$+1$	1	$-1, +1$
1	-1	2	$+1, -1$
2	$+1$	1	$+1, -1$
2	-1	0	$-1, +1$

The machine initially starts at state 0.

(a) (2 pts) After encoding k data bits $b_1, \ldots, b_k$, we would like to ensure that the machine returns to the initial state 0 by appending L termination bits $b_{k+1}, \ldots b_{k+L}$ to the data sequence. What is the value of L needed to ensure this? (Note that $(b_{k+1}, \ldots, b_{k+L})$ can depend on $(b_1, \ldots, b_k)$, but L can not.)

This encoding device, with the termination scheme in (a), is used as a transmitter for an AWGN channel. The channel output $y_1, y_2, \ldots$ is given by $y_i = \sqrt{\mathcal{E}_s} x_i + Z_i$, where $Z_1, Z_2, \ldots$ are i.i.d. $\mathcal{N}(0, \sigma^2)$.

(b) (2 pts) Draw a trellis diagram and explain how the receiver can implement the ML rule to produce $\hat{b}_1, \ldots, \hat{b}_k$ from $y_1, \ldots, y_n$.

(c) (3 pts) For $k = 3$, and $y_1, y_2, \ldots$ given by

$$+0.3, -1.1, +0.6, +0.9, -0.7, +1.0, +1.2, -1.1, -0.6, +0.7, +0.4, -1.2, +0.5, -0.5, \ldots$$

determine the maximally likely $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$. (You may not need the last few y_i's to do this.)

(d) (3 pts) For the all $+1$ reference path sketch the detour flow graph labeled with $I^i D^d$'s. Construct a system of equations of the form

$$\begin{aligned} A_1(I, D) &= ?? + ??\, A_1(I, D) + ??\, A_2(I, D) \\ A_2(I, D) &= ?? + ??\, A_1(I, D) + ??\, A_2(I, D) \\ A(I, D) &= ??\, A_2(I, D), \end{aligned}$$

where A_1 and A_2 denote the transfer functions until states 1 and 2, and verify that $A(I, D) = I^3 D^4 / (1 - D - I D^2)$.

(e) (2 pts) Differentiate $A(I, D)$ with respect to I, and use it to find an upper bound to the bit error probability on sending the all $+1$ sequence, as a function of $\mathcal{E}_s / \sigma^2$.