ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 35 Principles of Digital Communications
Final exam June 21, 2022

4 problems, 18 sub-problems each worth 4 points.
180 minutes
2 sheet (4 pages) of notes allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS.

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET.

Problem 1. Answer to the following question by true or false with a justification

(a) In a binary hypothesis testing setup with equally likely hypotheses we have designed decision regions R_0 and R_1 using the MAP rule. We find that $P(R_0|H_1) = P(R_1|H_0)$. Suppose now the hypotheses are not equally likely. Is the error probability of the new MAP rule higher than the equally likely case?

(b) In a m-ary hypothesis testing setup let $p_i(y)$ denote the distribution of the observation Y under hypothesis i ($i \in \{0, 1, \ldots, m-1\}$). Let $L_i(y) = \frac{p_i(y)}{p_{i-1}(y)}$ for $i \in \{1, \ldots, m-1\}$. Is $[L_1(Y), L_2(Y), \ldots, L_{m-1}(Y)]^T$ a sufficient statistic ?

(c) Suppose a communication system designed for m equally likely messages, with waveform set $w_0, \ldots, w_{m-1}$, and used over an AWGN channel of power spectral density $N_0/2$. The receiver uses the MAP rule. We are given the value of N_0, and the $m \times m$ matrix A with entries $A_{ij} = \langle w_i, w_j \rangle$. Is this information sufficient to compute the error probability?

(d) Suppose $\psi(t)$ is a Nyquist pulse with parameter T, i.e., with ψ_F denoting the Fourier Transform of ψ, $\sum_n \left|\psi_F\left(f - \frac{n}{T}\right)\right|^2 = T$. Is ψ a Nyquist pulse with parameter $2T$?

PROBLEM 2. Consider a communication system designed for an AWGN of power spectral density $N_0/2$, with two codewords c_0, c_1, both in $\mathbb{R}^n$, with $c_0 = -c_1 = [A, ..., A]^T$ for some $A > 0$.

(a) Assuming equally likely messages, what is the error probability P_e (in terms of A, n, N_0)?

We now use the communication system on a new AWGN of power spectral density is $N_0'/2$, where $N_0' = 4N_0$.

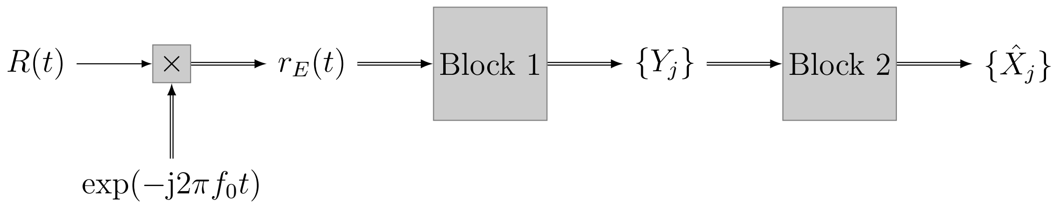
(b) We are allowed to change A to a new value A' but not allowed to change n. How should we choose A' to ensure that the error probability is not changed?

(c) Same as (b), but this time we are allowed to change n to a new value n' but not allowed to change A. How should we choose n' to ensure that the error probability is not changed.

(d) Suppose that the P_e found in (a) is very small. Suppose N_0 has been changed to $\tilde{N}_0 = N_0/4$. (n and A are not changed). Give a simple good estimate of the new error probablity $\tilde{P}_e$ in terms of P_e.

[Hint: $Q(x) \approx \exp(-x^2/2)$ for large $x > 0$]

Problem 3. Consider a 4-PSK passband communication system designed with the 'standard method': the transmitter accepts a sequence of i.i.d. and equally likely data symbols $\{X_j\}$ with X_j in $\{+\sqrt{\mathcal{E}_s}, -\sqrt{\mathcal{E}_s}, \mathrm{j}\sqrt{\mathcal{E}_s}, -\mathrm{j}\sqrt{\mathcal{E}_s}\}$ and produces a complex baseband waveform $w_E(t) = \sum_j X_j \psi(t-j)$, where ψ is a unit norm pulse orthogonal to its integer shifts, and has bandwidth B, (that is, $\psi_F(f) = 0$ for $|f| > B$). $w_E(t)$ is converted to a passband signal $w(t) = \sqrt{2}Re\{w_E(t)\exp(\mathrm{j}2\pi f_0 t)\}$ with $f_0 > B$, which is transmitted over an AWGN of power spectral density $N_0/2$.



(a) How does the receiver form $\{Y_j\}$ from the down-converted waveform $r_E(t)$? I.e., explain what should happen in Block 1, and what should happen in Block 2. Sketch or describe the MAP decision regions that estimate $\hat{X}_j$ from Y_j.

[Hint: consider using a matched filter]

Unfortunately there is a phase shift of θ in $[-\pi, \pi)$ in receiver's oscillator so it forms $r_E(t) = \sqrt{2}R(t)\exp\big(-\mathrm{j}(2\pi f_0 t + \theta)\big)$, instead of the $\sqrt{2}R(t)\exp(-\mathrm{j}2\pi f_0 t)$. The receiver is unaware of this phase shift and Block 1 and Block 2 do not take θ into account

(b) What is the error probability of the system (as designed in (a)) if $|\theta| \leq \frac{\pi}{4}$.

(c) How would you expect the system to perform if $|\theta| > \pi/4$?

(d) What is the minimum possible value of B? What is the corresponding pulse shape ψ with $\psi(0) = 1$?

PROBLEM 4. Consider a communication system where the message i in $\{0, \ldots, m-1\}$ is transmitted as a vector c_i in $\{+1, -1\}^n$. The transmitted vector c is received as $Y = c + Z$ where Z in $\mathbb{R}^n$ has *i.i.d.* components $Z_1, \ldots, Z_n$ with pdf $f_{Z_1}(z_1) = \exp(-|z_1|/\sigma)/(2\sigma)$), this is known as a Lapacian random variable.

(a) Assuming equally likely messages, find the optimal decision rule to be employed at the receiver.

(b) Noting that $|u - v| = |u| + |v| - 2\min(|u|, |v|)\mathbb{1}(uv > 0)$ show that the rule in (a) can be implemented as

$$\underset{i}{\operatorname{argmax}} \sum_j \min(1, |y_j|)\mathbb{1}(c_{ij} y_j > 0).$$

(c) Noting that $\frac{1}{2\sigma}\int_{-\infty}^{\infty} \exp\left(-\frac{|y-1|+|y+1|}{2\sigma}\right) dy = \left(1 + \frac{1}{\sigma}\right)\exp\left(-\frac{1}{\sigma}\right) =: g(\sigma)$, use the Union-Bhattacharyya bound to show that the error probability is upper bounded by

$$\Pr(error|i) \le \sum_{\ell : i \ne \ell} g(\sigma)^{d_H(c_i, c_\ell)}$$

where $d_H(a, b)$ is the number of places the vectors a and b differ.

(d) Consider now the rate 1/2 convolutional code with $x_{2j} = b_j$, $x_{2j-1} = b_j b_{j-1}$ for input bits b_i in $\{-1, 1\}$. Draw the detour graph of this code, making sure to label each branch with $D^\alpha I^\beta$.

(e) What is the branch metric $d(x_j, y_j)$ that should be used by Viterbi decoder that implements the MAP decoder?

(f) Find the transfer function $T(D, I)$ of the detour flow graph of the convolutional encoder described in (d). Using this find an upper bound to the bit error probability of this code.

[Hint: your answer should include the function $g(\sigma)$ of part (c).]